

Mathematics 2019 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

General Instructions :

(i) **All** questions are compulsory.

(ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of 4 questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

SECTION–A

1. If A is a square matrix satisfying A′ A = I, write the value of | A |. **[1]**

Solution : Given, A′A = I

Then $|A'A| = |I|$

$\Rightarrow |A'||A| = |I|$

$\Rightarrow |A||A| = |I|$ $[\because |A'| = |A|]$

$\Rightarrow |A|^2 = 1$ $[\because |I| = 1]$

$\Rightarrow |A| = \pm 1$ **Ans.**

2. If $y = x\,|x|$, find $\frac{dy}{dx}$ for $x < 0$. **[1]**

Solution : If $y = x\,|x|$

Then,

$$y = \begin{cases} -x^2 & x < 0 \\ x^2 & x > 0 \end{cases}$$

$\Rightarrow \frac{dy}{dx} = -2x$ when $x < 0$ **Ans.**

3. Find the order and degree (if defined) of the differential equation

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2 \log\left(\frac{d^2y}{dx^2}\right).$$ **[1]**

Solution :

$$\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 2x^2 \log\left(\frac{d^2y}{dx^2}\right)$$

Order of this equation is 2.

Degree of this equation is not defined. **Ans.**

4. Find the direction consines of a line which makes equal angles with the coordinate axes. [1]

Solution : Let the direction cosines of the line make an angle α with each of the coordinate axes and direction cosines be l, m and n.

$\therefore l = \cos\alpha, \quad m = \cos\alpha$ and $n = \cos\alpha$

$l^2 + m^2 + n^2 = 1$

$\therefore \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow 3\cos^2\alpha = 1$

$\Rightarrow \cos^2\alpha = \frac{1}{3}$

$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$

The direction cosines are $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

Ans.

OR

A line passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction of the vector $\hat{i} + \hat{j} - 2\hat{k}$. Find the equation of the line in cartesian form.

Solution :

The line passes through a point (2, – 1, 4) and has direction ratios proportional to (1, 1, – 2).

Cartesian equation of the line

$$= \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

$$= \frac{x - 2}{1} = \frac{y + 1}{1} = \frac{z - 4}{-2}$$ **Ans.**

SECTION–B

5. Examine whether the operation * defined on R, the set of all real numbers, by $a * b = \sqrt{a^2 + b^2}$ is a binary operation or not, and if it is a binary operation, find whether it is associative or not.** **[2]**

** **Answer is not given due to the change in present syllabus.**

6. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$, show that $(A-2I)(A-3I) = 0$. **[2]**

Solution : Given,

$$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

Then, $(A-2I)(A-3I)$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$ **Hence Proved**

7. **Find :** $\int \sqrt{3-2x-x^2}\,dx$. **[2]**

Solution : Let $I = \int \sqrt{3-2x-x^2}\,dx$

$$= \int \sqrt{-(x^2+2x-3)}\,dx$$

$$= \int \sqrt{-(x^2+1+2x-3-1)}\,dx$$

$$= \int \sqrt{-[(x+1)^2-4]}\,dx$$

$$= \int \sqrt{4-(x+1)^2}\,dx$$

$$= \int \sqrt{2^2-(x+1)^2}\,dx$$

$$\left[\because \int \sqrt{a^2-x^2}\,dx = x\sqrt{\frac{a^2-x^2}{2}} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \frac{1}{2}(x+1)\sqrt{4-(x+1)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x+1}{2}\right) + c$$

$$\therefore \quad I = \int \sqrt{2^2-(x+1)^2}\,dx$$

$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2\sin^{-1}\frac{(x+1)}{2} + c$$ **Ans.**

8. **Find :**

$$\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\,dx.$$ **[2]**

Solution :

Let $I = \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x.\cos^2 x}\,dx$

$$= \int \frac{\sin^3 x\,dx}{\sin^2 x.\cos^2 x} + \int \frac{\cos^3 x}{\sin^2 x.\cos^2 x}\,dx$$

$$= \int \frac{\sin x}{\cos^2 x}\,dx + \int \frac{\cos x}{\sin^2 x}\,dx$$

$$= \int \tan x.\sec x\,dx + \int \cot x.\text{cosec}\,x\,dx$$

$$= \sec x - \text{cosec}\,x + c$$

$$\left[\because \int \tan x.\sec x\,dx = \sec x \text{ and } \int \cot x.\text{cosec}\,x\,dx = -\text{cosec}\,x\right]$$

Ans.

OR

Find : $\int \frac{x-3}{(x-1)^3} e^x\,dx$

Solution :

Let, $I = \int \frac{x-3}{(x-1)^3} e^x\,dx$

$$= \int \frac{x-2-1}{(x-1)^3} e^x\,dx$$

$$= \int \frac{x-1-2}{(x-1)^3} e^x\,dx$$

$$= \int \left[\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3}\right] e^x\,dx$$

$$= \int \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right] e^x\,dx \quad \text{...(i)}$$

Here, $\frac{d}{dx}\left[\frac{1}{(x-1)^2}\right] = \frac{d}{dx}(x-1)^{-2} = -2(x-1)^{-3}$

Then, $f'(x) = \frac{-2}{(x-1)^3}$

We know that,

$$\int [f(x)+f'(x)]\,e^x\,dx = e^x f(x) + c$$

Then from equation (i), we have

$$\int \left[\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right] e^x\,dx = \frac{e^x}{(x-1)^2} + c$$ **Ans.**

9. Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants. [2]

Solution : Given, $y = Ae^{2x} + Be^{-2x}$ (i)

On differentiating equation (i) w.r.t. x, we get

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x} \quad ...(ii)$$

Again, differentiating equation (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 4Ae^{2x} - 4Be^{-2x}$$

$$= 4(Ae^{2x} + Be^{-2x})$$

$$= 4y$$

$\therefore \frac{d^2y}{dx^2} - 4y = 0$ is the required differential equation **Ans.**

10. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $\vec{a} \times \vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$. [2]

Solution : Given,

$$|\vec{a}| = 2, |\vec{b}| = 7 \text{ and } \left(\vec{a} \times \vec{b}\right) = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$\therefore$ Angle between $\vec{a}$ and $\vec{b}$ is given by

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|} \quad ...(i)$$

Then, $|\vec{a} \times \vec{b}| = \sqrt{(3)^2 + (2)^2 + (6)^2}$

$$= \sqrt{49} = 7$$

$\therefore \quad \sin\theta = \frac{7}{2 \times 7} = \frac{1}{2}$

$\Rightarrow \quad \sin\theta = \sin\frac{\pi}{6}$

$\Rightarrow \quad \theta = \frac{\pi}{6}$ **Ans.**

OR

Find the volume of a cuboid whose edges are given by $-3\hat{i} + 7\hat{j} + 5\hat{k}$, $-5\hat{i} + 7\hat{j} - 3\hat{k}$ and $7\hat{i} - 5\hat{j} - 3\hat{k}$.

Solution :

If a, b, c are edges of a cuboid.

Then, volume of cuboid $= \vec{a}.(\vec{b} \times \vec{c})$

Here,

$$\vec{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}$$

$$\vec{b} = -5\hat{i} + 7\hat{j} - 3\hat{k}$$

$$\vec{c} = 7\hat{i} + 5\hat{j} - 3\hat{k}$$

Then,

$$\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$$= -3(-21 - 15) - 7(15 + 21) + 5(25 - 49)$$

$$= -3 \times (-36) - 7 \times 36 + 5 \times (-24)$$

$$= 108 - 252 - 120$$

$$= -264 \text{ cubic units}$$ **Ans.**

11. If P(not A) = 0·7, P(B) = 0·7 and P(B/A) = 0·5, then find P(A/B). [2]

Solution :

Given, P(not A) = 0.7

P (B) = 0.7

and P (B/A) = 0.5

We know that

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$0.5 = \frac{P(A \cap B)}{0.3}$$

[$\because$ P(not A) = 0.7 then P(A) = 1 − 0.7 = 0.3]

$\Rightarrow \quad P(A \cap B) = 0.15$

Also we know,

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{0.15}{0.7} = \frac{15}{7}$$ **Ans.**

12. A coin is tossed 5 times. What is the probability of getting (i) 3 heads, (ii) at most 3 heads? [2]

Solution :

Here, $n = 5$, $p = \frac{1}{2}$ and $q = \frac{1}{2}$

We know that,

$$p(x) = {}^nC_x p^x q^{n-x}$$

$$= {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}$$

(i) For 3 heads, $x = 3$

Then, the probability of getting 3 heads is

$$P(3) = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^{5-3}$$

$$= {}^5C_3 \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^2$$

$$= {}^5C_3 \times \left(\frac{1}{2}\right)^5$$

$$= \frac{5\times4\times!3}{2\times!3}\left(\frac{1}{2}\right)^5$$

$$= 10\left(\frac{1}{2}\right)^5 = \frac{5}{16}$$ **Ans.**

(ii) Probability of getting at most 3 heads is

$P(x \le 3) = P(0) + P(1) + P(2) + P(3)$

Then, $P(0) = {}^5C_0\left(\frac{1}{2}\right)^5$

$$= \frac{1}{32}$$

$$P(1) = {}^5C_1\left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

$$P(2) = {}^5C_2\left(\frac{1}{2}\right)^5 = \frac{5}{16}$$

$$P(3) = {}^5C_3\left(\frac{1}{2}\right)^5 = \frac{5}{16}$$

$\therefore$ $$P(x \le 3) = \frac{1}{32}+\frac{5}{32}+\frac{5}{16}+\frac{5}{16}$$

$$= \frac{26}{32} = \frac{13}{16}$$ **Ans.**

OR

Find the probabilty distribution of X, the number of heads in a simultaneous toss of two coins.

Solution :

If we toss two coins simultaneously then sample space is given by (HH, HT, TH, TT)

Then probability distribution is,

X(No. of heads)	0	1	2
$P(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Ans.

SECTION–C

13. Check whether the relation R defined on the set A = {1, 2, 3, 4, 5, 6} as R = {(*a*, *b*) : *b* = *a* + 1} is reflexive, symmetric or transitive. [4]

Solution : Here, $R = \{(a, b) : b = a + 1\}$

$\therefore$ $R = \{(a, a+1) : a, a+1 \in (1, 2, 3, 4, 5, 6)\}$

$\Rightarrow$ $R = \{(1,2,) (2,3), (3,4), (4,5), (5,6)\}$

(i) R is not reflexive as $(a, a) \notin R \forall a$

(ii) R is not symmetric as $(1, 2) \in R$ but $(2, 1) \notin R$

(iii) R is not transitive as $(1, 2) \in R$, $(2, 3) \in R$ but $(1, 3) \notin R$

OR

Let $f : N \to Y$ be a function defined as

$$f(x) = 4x + 3,$$

where $Y = \{y \in N : y = 4x + 3, \text{ for some } x \in N\}$. Show that f is invertible. Find its inverse.

Solution : Consider an arbitrary element of Y. By the definition of y, $y = 4x + 3$, for some x in the domain N.

This shows that $x = \frac{y-3}{4}$

Define $g : Y \to N$ by $g(y) = \frac{y-3}{4}$

Now, $gof(x) = g(f(x)) = g(4x+3) = \frac{4x+3-3}{4} = x$

and $fog(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = \frac{4(y-3)}{4} + 3$

$$= y - 3 + 3 = y$$

This shows that $gof = I_N$ and $fog = I_y$ which implies that f is invertible and g is the inverse of f.

Hence Proved.

$\therefore$ Inverse of $f = g(y) = \frac{y-3}{4}$ **Ans.**

14. Find the value of $\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$. [4]

Solution :

$\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$

$$= \sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$$

$$= \sin\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)$$

$$\left(\because \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}\right)$$

$$= \sin\left(\tan^{-1}\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \sin\left(\tan^{-1}\frac{\frac{9+8}{12}}{1-\frac{6}{12}}\right)$$

$$= \sin\left(\tan^{-1}\frac{\frac{17}{12}}{\frac{6}{12}}\right)$$

$$= \sin(\tan^{-1}17/16)$$

$$= \sin\left(\tan^{-1}\frac{17}{6}\right)$$

$$= \sin\left(\sin^{-1}\frac{17}{\sqrt{325}}\right)$$

$$\left(\because \tan^{-1}\frac{17}{6} = \sin^{-1}\frac{17}{\sqrt{325}}\right)$$

$$= \frac{17}{\sqrt{325}}$$ **Ans.**

15. Using properties of determinants, show that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$$

[4]

Solution :

$$\text{L.H.S.} = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

On applying operation $C_1 \to C_1 + C_2 + C_3$, we have

$$= \begin{vmatrix} 3a-a+b-a+c & -a+b & -a+c \\ -b+a+3b-b+c & 3b & -b+c \\ -c+a-c+b+3c & -c+b & 3c \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Again, applying operations $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we get

$$= (a+b+c)\begin{vmatrix} 0 & -a+c & -a-2c \\ 0 & 2b+c & -b-2c \\ 1 & -c+b & 3c \end{vmatrix}$$

On applying $R_2 \to R_2 - R_1$, we get

$$= (a+b+c)\begin{vmatrix} 0 & -a+c & -a-2c \\ 0 & 2b+a & -b+a \\ 1 & -c+b & 3c \end{vmatrix}$$

Expanding about C_1, we get

$$= (a+b+c)\begin{vmatrix} -a+c & -(a+2c) \\ 2b+a & a-b \end{vmatrix}$$

$= (a+b+c)\,[(c-a)(a-b) + (a+2c)(2b+a)]$

$= (a+b+c)\,[ac - bc - a^2 + ab + 2ab + a^2 + 4bc + 2ac]$

$= (a+b+c)\,[3ac + 3bc + 3ab]$

$= (a+b+c) \times 3(ac+bc+ab)$

$= 3(a+b+c)(ac+bc+ab) =$ R. H. S.

Hence Proved.

16. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and $x \neq y$, prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$. **[4]**

Solution :

Given, $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x}$

On squaring both sides, we get

$\Rightarrow \quad x^2(1+y) = y^2(1+x)$

$\Rightarrow \quad x^2 + x^2y = y^2 + y^2x$

$\Rightarrow \quad x^2 + x^2y - y^2 - y^2x = 0$

$\Rightarrow \quad x^2 - y^2 + x^2y - y^2x = 0$

$\Rightarrow \quad x^2 - y^2 + x^2y - y^2x = 0$

$\Rightarrow (x-y)(x+y) + xy(x-y) = 0$

$\Rightarrow \quad (x-y)[x+y+xy] = 0$

$\Rightarrow \quad x + y + xy = 0 \qquad [\because x \neq y]$

$\Rightarrow \quad y = -\frac{x}{1+x} \qquad$ ...(i)

On differentiating equation (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+x)\frac{d(-x)}{dx} - (-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)(-1) + x(1)}{(1+x)^2}$$

$$= \frac{-1-x+x}{(1+x)^2}$$

$$= \frac{-1}{(1+x)^2}$$

$\therefore \quad$ L.H.S = R.H.S. **Hence Proved.**

OR

If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.

Solution :

Given, $(\cos x)^y = (\sin y)^x$

On taking log on both sides, we get

$y \log(\cos x) = x \log(\sin y) \qquad$ ..(i)

On differentiating equation (i) w.r.t. x,

$$\Rightarrow y \times \frac{d}{dx}(\log \cos x) + \log \cos x \frac{dy}{dx}$$

$$= \log \sin y \frac{dx}{dx} + x\frac{d}{dx}(\log \sin y)$$

$$\Rightarrow \; y \times \frac{1}{\cos x}(-\sin x) + \log \cos x \frac{dy}{dx}$$

$$= \log \sin y + x \cdot \frac{\cos y}{\sin y} \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad -y\tan x + \frac{dy}{dx}\log\cos x = \log\sin y + x\cot y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}(\log\cos x - x\cot y) = \log\sin y + y\tan x$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log\sin y + y\tan x}{\log\cos x - x\cot y}$$ **Ans.**

17. If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that $\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ is a constant independent of a and b. **[4]**

Solution :

If $\quad (x-a)^2 + (y-b)^2 = c^2, \qquad c > 0 \qquad$ ...(i)

On differentiating equation (i) w.r.t. x, we get

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x - a + (y-b)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(x-a)}{(y-b)} \qquad \text{...(ii)}$$

Again, differentiating equation (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{(y-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b) - (x-a)\,dy/dx}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b) - (x-a)\,dy/dx}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b) + \frac{(x-a)(x-a)}{(y-b)}}{(y-b)^2}\right]$$

$$\left[\text{from equation (ii), } \frac{dy}{dx} = -\frac{(x-a)}{(y-b)}\right]$$

$$= -\left[\frac{(y-b)^2 + \frac{(x-a)^2}{(y-b)}}{(y-b)^2}\right]$$

$$= -\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3}\right]$$

$$= -\left[\frac{c^2}{(y-b)^3}\right]$$

$[\because$ From equation (i), $(x-a)^2 + (y-b)^2 = c^2]$

Now, $\quad \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$$= \frac{-\left[1 + \frac{(x-a)^2}{(y-b)^2}\right]^{3/2}}{\frac{c^2}{(y-b)^3}}$$

$$= -\frac{\left[(y-b)^2 + (x-a)^2\right]^{3/2} \times (y-b)^2}{(y-b)^{2\times 3/2} \times c^2}$$

$$= -\frac{\left[(y-b)^2 + (x-a)^2\right]^{3/2} \times (y-b)^2}{(y-b)^2 \times c^2}$$

$$= -\frac{\left[(y-b)^2 + (x-a)^2\right]^{3/2}}{c^2}$$

$$= -\frac{-c^{2\times 3/2}}{c^2}$$

$$\left[\because c^2 = (y-b)^2 + (x-a)^2\right]$$

$$= \frac{-c^3}{c^2} = -c = \text{constant}$$

It shows that $\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ is independent of a and b

Hence Proved.

18. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $(-1, 4)$. [4]

Solution :

Suppose the normal at $P(x_1, y_1)$ on the parabola $x^2 = 4y$ passes through $(-1, 4)$

Since, $P(x_1, y_1)$ lies on $x^2 = 4y$

$$\therefore \quad x_1^2 = 4y_1 \qquad \text{...(i)}$$

The equation of curve is $x^2 = 4y$

Differentiating with respect to x, we have

$$2x = 4\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{x_1}{2}$$

The equation (x_1, y_1) of normal at $P(x_1, y_1)$ is

$$y - y_1 = \frac{-1}{\frac{dy}{dx}}(x - x_1)$$

$$y - y_1 = \frac{-2}{x_1}(x - x_1) \quad \text{...(ii)}$$

$\because$ It passes through $(-1, 4)$,

$\therefore$ Putting $x = -1$ and $y = y$, we get

$$4 - y_1 = \frac{-2}{x_1}(-1 - x_1)$$

$$\Rightarrow \quad 4 - y_1 = \frac{2}{x_1}(1 + x_1)$$

$$\Rightarrow \quad 4x_1 - x_1y_1 = 2 + 2x_1$$

$$\Rightarrow \quad 2x_1 = 2 + x_1y_1$$

$$\Rightarrow \quad \frac{2x_1 - 2}{x_1} = y_1 \quad \text{....(iii)}$$

Eliminating y_1 from equation (i), we have

$$x_1^2 = 4\left(\frac{2x_1 - 2}{x_1}\right)$$

$$x_1^3 = 8x_1 - 8$$

$$\Rightarrow \quad x_1 = 2$$

Putting $x_1 = 2$ in (iii), we get $y_1 = 1$

Putting values of x_1, y_1 in (ii), we get

$$y - 1 = -1(x - 2)$$

$$x + y - 3 = 0$$

Which is the required equation of normal to the given curve. **Ans.**

19. Find : $\int \frac{x^2 + x + 1}{(x+2)(x^2+1)} dx.$ **[4]**

Solution :

Let, $$I = \int \frac{x^2 + x + 1}{(x+2)(x^2+1)} dx$$

By partial fractions

$$\frac{x^2 + x + 1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx + C}{x^2 + 1}$$

$$x^2 + x + 1 = A(x^2 + 1) + Bx + C(x + 2)$$

$$= Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$$

$$x^2 + x + 1 = x^2(A + B) + x(2B + C) + A + 2C \quad \text{...(i)}$$

On comparing coefficients of equation (i), we get

$$1 = A + B$$

and $$1 = 2B + C$$

On putting $x = -2$ in (i), we get

$$(-2)^2 + (-2) + 1 = (-2)^2 A + A$$

$$3 = 4A + A$$

$$\frac{3}{5} = A$$

Then, $$1 = \frac{3}{5} + B$$

$$B = \frac{2}{5} \text{ and } C = \frac{1}{5}$$

Hence, $$\frac{x^2 + x + 1}{(x+2)(x^2+1)} = \frac{3}{5(x+2)} + \frac{\frac{2}{5}x + \frac{1}{5}}{x^2 + 1}$$

Then,

$$\int \frac{(x^2 + x + 1)dx}{(x+2)(x^2+1)} = \frac{3}{5}\int \frac{1}{x+2} dx + \frac{2}{5}\int \frac{x}{x^2+1} dx + \frac{1}{5}\int \frac{1}{(x^2+1)} dx$$

$$= \frac{3}{5} \log |x + 2| + \frac{1}{5}\int \frac{2x}{x^2+1} + \frac{1}{5}\int \frac{dx}{x^2+1}$$

$$= \frac{3}{5} \log |x + 2| + \frac{1}{5} \log |x^2 + 1| + \frac{1}{5} \tan^{-1} x + c$$

Ans.

20. Prove that : $\int_0^a f(x)dx = \int_0^a f(a - x)dx$ **and hence evaluate** $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx.$ **[4]**

Solution :

$$\text{R.H.S.} = \int_0^a f(a - x)\, dx$$

Let $$a - x = v$$

$$-1 = \frac{dv}{dx}, \text{ for } x = 0, v = a$$

$$x = a, v = 0$$

Then,

$$\text{R.H.S.} = \int_a^0 f(v)(-dv)$$

$$= -\int_a^0 f(v)(dv)$$

$$= \int_0^a f(v)(dv)$$

$$\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)dx\right]$$

Now, replacing v by x,

$$= \int_0^a f(x)\,dx$$

Hence, $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$ **Hence Proved.**

Now, Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$ ...(i)

$$I = \int_a^{\pi/2} \frac{\frac{\pi}{2} - x}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\cos x + \sin x} dx \qquad \text{(ii)}$$

Adding equations (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{x + \frac{\pi}{2} - x}{\sin x + \cos x} dx$$

$$2I = \int_0^{\pi/2} \frac{\frac{\pi}{2}}{a \sin x + \cos x} dx$$

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\frac{2\tan x/2}{1 + \tan^2 x/2} + \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}} dx$$

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1 + \tan^2 \frac{x}{2}}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}} dx$$

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sec^2 x/2}{-\tan^2 x/2 + 2\tan x/2 + 1} dx$$

Let, $\tan\frac{x}{2} = t$

Then, $\frac{d}{dx}\left(\tan\frac{x}{2}\right) = \frac{d}{dx}(t)$

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$$

$$\sec^2\frac{x}{a}dx = 2dt$$

Also, $x = 0$

$\Rightarrow \quad t = \tan 0° = 0$

and $\quad x = \pi/2$

$\Rightarrow \quad t = \tan \pi/4 = 1$

$$\therefore \quad 2I = \frac{\pi}{2}\int_0^1 \frac{2dt}{-t^2 + 2t + 1}$$

$$= \pi\int_0^1 \frac{dt}{-t^2 + 2t + 1}$$

$$= \pi\int_0^1 \frac{dt}{-(t^2 - 2t - 1)}$$

$$= \pi\int_0^1 \frac{dt}{-\left[(t-1)^2 - 2\right]}$$

$$= \pi\int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2}$$

$$= \pi \times \frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2} + t - 1}{\sqrt{2} - t + 1}\right|\right]_0^1$$

$$= \pi \times \frac{1}{2\sqrt{2}}\left[\log 1 - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= \frac{-\pi}{2\sqrt{2}}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) = \frac{\pi}{2\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

$$\Rightarrow \quad 2I = \frac{\pi}{2\sqrt{2}}\log\left\{\frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)}\right\}$$

$$\Rightarrow \quad 2I = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)^2$$

$$\Rightarrow \quad I = \frac{\pi}{4\sqrt{2}}\log(\sqrt{2}+1)^2$$

$$\Rightarrow \quad I = \frac{2\pi}{4\sqrt{2}}\log(\sqrt{2}+1)$$

$$\Rightarrow \quad I = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$ **Ans.**

21. Solve the differential equation :

$$x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$$ **[4]**

Solution :

Given, $x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$

Let, $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad x\frac{dy}{dx} = y - x\tan\left(\frac{y}{x}\right)$

$$\Rightarrow \quad x\left(v+x\frac{dv}{dx}\right)=vx-x\tan\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad x\left(v+x\frac{dv}{dx}\right)=x(v-\tan v)$$

$$\Rightarrow \quad xv+x^2\frac{dv}{dx}=xv-x\tan v$$

$$\Rightarrow \quad x^2\frac{dv}{dx}=-x\tan v$$

$$\Rightarrow \quad x\frac{dv}{dx}=-\tan v$$

$$\Rightarrow \quad \frac{dv}{\tan v}=-\frac{dx}{x}$$

Then $\int \cot v\, dv=-\int\frac{dx}{x}$

$$\log\sin v=-\log x+\log c$$

$$\Rightarrow \quad \log\sin v=\log\frac{c}{x}$$

$$\Rightarrow \quad \sin\frac{y}{x}=\frac{c}{x}$$

$$\Rightarrow \quad x\sin\frac{y}{x}=c$$

Ans.

OR

Solve the differential equation :

$$\frac{dy}{dx}=-\left[\frac{x+y\cos x}{1+\sin x}\right]$$

Solution :

$$\frac{dy}{dx}=-\left[\frac{x+y\cos x}{1+\sin x}\right]$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{x}{1+\sin x}-\frac{y\cos x}{1+\sin x}$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y\cos x}{1+\sin x}=\frac{-x}{1+\sin x}$$

Here , $P=\dfrac{\cos x}{1+\sin x}$ and $Q=\dfrac{-x}{1+\sin x}$

Then, $IF=e^{\int pdx}$

$$=e^{\int\frac{\cos x}{1+\sin x}dx}$$

$$=e^{\log|1+\sin x|}$$

$$=1+\sin x$$

Then, $y\times IF=\int Q\times IF\ dx+c$

$$y\,(1+\sin x)=\int\frac{-x}{1+\sin x}\times(1+\sin x)dx$$

$$=\int -x\,dx$$

$$y\,(1+\sin x)=\frac{-x^2}{2}+c$$

Ans.

22. **The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of the vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$.** **[4]**

Solution :

Here, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\hat{n}$ is unit vector

$$\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$$

$$\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$$

$$\vec{b}+\vec{c}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

Then, $\hat{n}=\dfrac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+36+4}}$

Given, $\vec{a}.\hat{n}=1$

$$(\hat{i}+\hat{j}+\hat{k}).\left(\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+40}}\right)=1$$

$$\Rightarrow \quad (2+\lambda)+6-2=\sqrt{(2+\lambda)^2+40}$$

$$\left[\because \hat{i}\cdot\hat{i}=1,\ \hat{j}\cdot\hat{j}=1,\ \hat{k}\cdot\hat{k}=1\right]$$

$$\Rightarrow \quad (2+\lambda)+4=\sqrt{(2+\lambda)^2+40}$$

$$\Rightarrow \quad \lambda+6=\sqrt{(2+\lambda)^2+40}$$

On squaring both sides, we get

$$(6+\lambda)^2=(2+\lambda)^2+40$$

$$\Rightarrow \quad 36+\lambda^2+12\lambda=4+\lambda^2+4\lambda+40$$

$$\Rightarrow 36+12\lambda-4-4\lambda-40=0$$

$$\Rightarrow \quad 8\lambda-8=0$$

$$\Rightarrow \quad \lambda=1$$

Then, $\vec{b}+\vec{c}=(2+\lambda)\hat{i}+6\hat{j}+-2\hat{k}$

$$=3\hat{i}+6\hat{j}-2\hat{k}$$

unit vector along $(\vec{b}+\vec{c})=\dfrac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{9+36+4}}$

$$=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{49}}$$

$$=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{7}$$

Ans.

23. If the lines $\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$ and $\frac{x-1}{3\lambda}=\frac{y-1}{2}=\frac{z-6}{-5}$ are perpendicular, find the value of λ. Hence find whether the lines are intersecting or not. **[4]**

Solution :

The equation of the given lines are,

$$\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2} \text{ and } \frac{x-1}{3\lambda}=\frac{y-1}{2}=\frac{z-6}{-5}$$

If these lines are perpendicular, then

$$a_1a_2+b_1b_2+c_1c_2=0$$

$$\Rightarrow \quad -3\times 3\lambda+2\lambda\times 2+2\times(-5) = 0$$

$$\Rightarrow \quad -9\lambda+4\lambda-10 = 0$$

$$\Rightarrow \quad -5\lambda = 10$$

$$\Rightarrow \quad \lambda = -2$$

Now, the lines are

$$\frac{x-1}{-3}=\frac{y-2}{-4}=\frac{z-3}{2} \text{ and } \frac{x-1}{-6}=\frac{y-1}{2}=\frac{z-6}{-5}$$

The co-ordinates of any point on first line are given by :

$$\frac{x-1}{-3}=\frac{y-2}{-4}=\frac{z-3}{2}=\alpha$$

or $\quad x-1 = -3\alpha \Rightarrow x=-3\alpha+1$

$y-2 = -4\alpha \Rightarrow y=-4\alpha+2$

$z-3 = 2\alpha \Rightarrow z=2\alpha+3$

So, coordinates of any point on this line are, $(-3\alpha+1, 4\alpha+2, 2\alpha+3)$.

The coordinates of any point on second line are given by :

$$\frac{x-1}{-6}=\frac{y-1}{2}=\frac{z-6}{-5}=\beta$$

or $\quad x-1 = -6\beta \Rightarrow x=-6\beta+1$

$y-1 = 2\beta \Rightarrow y=2\beta+1$

$z-6 = -5\beta \Rightarrow z=-5\beta+6$

So, co-ordinates of any point on second line are $(-6\beta+1, 2\beta+1, -5\beta+6)$.

If lines intersect then they have a common point. So, for some value of α and β, we have

$$-3\alpha+1 = -6\beta+1$$

$$\Rightarrow \quad -3\alpha = -6\beta$$

$$\Rightarrow \quad \alpha = 2\beta$$

and $\quad -4\alpha+2 = 2\beta+1$

$$\Rightarrow \quad -4\alpha+1 = 2\beta$$

On solving, we have

$$\alpha = \frac{1}{5}, \text{ and } \beta = \frac{1}{10}$$

The values of α and β do not satisfy the third equation. Hence, lines do not interecet each other. **Ans.**

SECTION–D

24. If $A=\begin{bmatrix}1 & 3 & 4\\ 2 & 1 & 2\\ 5 & 1 & 1\end{bmatrix}$, find A^{-1} **[6]**

Hence solve the system of equations

$$x+3y+4z = 8$$
$$2x+y+2z = 5$$
and $$5x+y+z = 7$$

Solution :

If $\quad A = \begin{bmatrix}1 & 3 & 4\\ 2 & 1 & 2\\ 5 & 1 & 1\end{bmatrix}$

$$|A| = 1(1-2)-3(2-10)+4(2-5)$$
$$= 11 \neq 0$$

Then cofactors of A are

$A_{11}=-1, A_{12}=8, A_{13}=-3, A_{21}=1, A_{22}=-19,$
$A_{23}=14, A_{31}=2, A_{32}=6$ and $A_{33}=-5$

$$\text{Then, adj } A = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A^{33}\end{bmatrix}^T$$

$$= \begin{bmatrix}-1 & 8 & -3\\ 1 & -19 & 14\\ 2 & 6 & -5\end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix}-1 & 1 & 2\\ 8 & -19 & 6\\ -3 & 14 & -5\end{bmatrix}$$

Now, $\quad A^{-1} = \frac{1}{|A|}\text{adj}A$

$$= \frac{1}{11}\begin{bmatrix}-1 & 1 & 2\\ 8 & -19 & 6\\ -3 & 14 & -5\end{bmatrix} \quad \text{...(ii) **Ans.**}$$

Given system of equations are

$$x+3y+4z = 8$$
$$2x+y+2z = 5$$
$$5x+y+z = 7$$

Let, $A = \begin{bmatrix}1 & 3 & 4\\ 2 & 1 & 2\\ 5 & 1 & 1\end{bmatrix}$ $B = \begin{bmatrix}8\\ 5\\ 7\end{bmatrix}$ and $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$

Then, $AX = B$

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$= \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

[Using (ii) equation]

$$= \frac{1}{11}\begin{bmatrix} -8+5+14 \\ 64-95+42 \\ -24+70-35 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix}$$

$$X = \frac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $x = 1$, $y = 1$, $z = 1$ **Ans.**

OR

Find the inverse of the following matrix, using elementary transformation :

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Solution :

Given, $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

We know that,

$$AA^{-1} = I$$

$$\therefore \quad A = IA$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

On applying $R_3 \to R_3 + 3R_1$, we get

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 6 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - R_2$, we get

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} A$$

Interchanging $R_1 \leftrightarrow R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 5R_1$ and $R_3 \to R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ -5 & 2 & -2 \end{bmatrix} A$$

Applying $R_3 \to (-1) R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

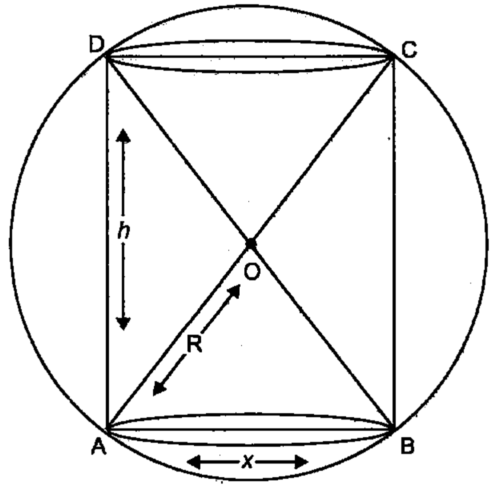
Hence, the required inverse of the matrix is

$$\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & 5 \\ 5 & -2 & 2 \end{bmatrix}$$ **Ans.**

25. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. **[6]**

Solution :

Let, 'x' be the diameter of the base of the cylinder and let 'h' be height of the cylinder.



In ΔABC, we have

$$(BC)^2 + (AB)^2 = (AC)^2$$

$$h^2 + x^2 = (2R)^2$$

$$x^2 = 4R^2 - h^2 \quad \text{...(i)}$$

Volume of cylinder, $V = \pi r^2 h$

$$\Rightarrow \quad V = \pi \times \left(\frac{x}{2}\right)^2 \times h$$

$$\Rightarrow \quad V = \pi \times \frac{x^2}{4} \times h$$

$$\Rightarrow \quad V = \frac{\pi(4R^2 - h^2)}{4} \times h \quad \text{[Using (i)]}$$

$$\Rightarrow \quad V = \frac{4\pi R^2 \times h}{4} - \frac{\pi h^3}{4}$$

$$\Rightarrow \quad V = \pi h R^2 - \pi \frac{h^3}{4} \quad \text{...(ii)}$$

On differentiating equation (ii) w.r.t. h, we get

$$\frac{dV}{dh} = \frac{d\left(\pi h R - \pi h^3/4\right)}{dh}$$

$$\Rightarrow \quad \frac{dV}{dh} = \pi R^2 \frac{d(h)}{dh} - \frac{\pi}{4}\frac{d(h^3)}{dh}$$

$$\Rightarrow \quad \frac{dV}{dh} = \pi R^2 - \frac{\pi}{4}(3h^2)$$

$$\Rightarrow \quad \frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4} \quad \text{...(iii)}$$

$$\Rightarrow \quad 0 = \pi R^2 - \frac{3\pi h^2}{4} \quad \left(\because \frac{dV}{dh} = 0\right)$$

$$\Rightarrow \quad h = \frac{2R}{\sqrt{3}}$$

Again, differentiating equation (iii) w.r.t. h, we get

$$\frac{d^2V}{dh^2} = \frac{d}{dh}\left(\pi R^2 - \frac{3\pi h^2}{4}\right)$$

$$\Rightarrow \quad \frac{d^2V}{dh^2} = 0 - \frac{3}{4}\pi \times 2h$$

$$\Rightarrow \quad \frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$$

At $h = \frac{2R}{\sqrt{3}}$, we have

$$\frac{d^2V}{dh^2} = \frac{-3\pi}{2}\left(\frac{2R}{\sqrt{3}}\right) = -\sqrt{3}\,\pi R$$

$$\frac{d^2V}{dh^2} < 0$$

Hence, $h = \frac{2R}{\sqrt{3}}$ is a point of maxima.

So, V is maximum when $h = \frac{2R}{\sqrt{3}}$

Hence, the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. **Hence Proved.**

From (i), we have

$$x^2 = 4R^2 - h^2$$

$$\Rightarrow \quad x^2 = 4R^2 - \left(\frac{2R}{\sqrt{3}}\right)^2$$

$$\Rightarrow \quad x^2 = \frac{8}{3}R^2$$

$\therefore$ Maximum Volume of cylinder

$$= \pi\left(\frac{x}{2}\right)^2 \times h$$

$$= \pi \times \frac{x^2}{4} \times h$$

$$= \frac{\pi}{4}x^2 h$$

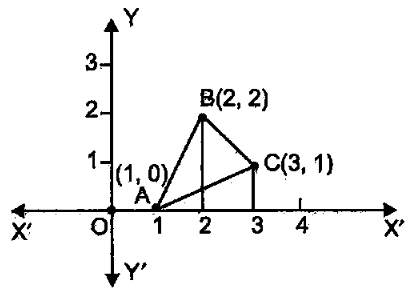
$$= \frac{\pi}{4} \times \frac{8R^2}{3} \times \frac{2R}{\sqrt{3}}$$

$$= \frac{4\pi R^3}{3\sqrt{3}} \quad \textbf{Ans.}$$

26. Using method of integration, find the area of the triangle whose vertices are (1, 0), (2, 2) and (3, 1). [6]

Solution :

A(1, 0), B(2, 2) and C(3, 1)



Let A(1, 0), B(2, 2) and C(3, 1) be the vertices of a triangle ABC.

Area of ΔABC = Area of ΔABD + Area of trapezium BDEC – Area of ΔAEC

Now, Equation of side AB,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = 2(x - 1)$$

$$y = 2(x - 1) \quad \text{...(i)}$$

Equation of line BC,

$$y - 2 = \frac{1-2}{3-2}(x - 2)$$

$$y - 2 = \frac{-1}{1}(x - 2)$$

$$y - 2 = -(x-2)$$
$$y = 2-(x-2)$$
$$y = 4 - x \qquad ...(ii)$$

Equation of line AC,

$$y - 0 = \frac{1-0}{3-1}(x-1)$$
$$y - 0 = \frac{1}{2}\ (x-1)$$
$$y = \frac{1}{2}(x-1) \qquad ...(iii)$$

Hence, area of ΔABC

$$= \int_1^2 2(x-1)dx + \int_2^3 (4-x)dx - \int_1^3 \frac{x-1}{2}dx$$

$$= 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$$

$$= 2\left[\left(\frac{2^2}{2}-2\right)-\left(\frac{1}{2}-1\right)+\left[\left(4\times 3-\frac{3^2}{2}\right)\right.\right.$$
$$\left.\left.-\left(4\times 2-\frac{2^2}{2}\right)\right]-\frac{1}{2}\left[\left(\frac{3^2}{2}-3\right)-\left(\frac{1}{2}-1\right)\right]\right.$$

$$= 2\left(\frac{1}{2}\right)+\frac{3}{2}-1 = \frac{3}{2} \text{ sq. units.} \qquad \textbf{Ans.}$$

OR

Using method of integration, find the area of the region enclosed between two circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

Solution :

Equations of the given circles are,

$$x^2 + y^2 = 4 \qquad ...(i)$$
$$(x-2)^2 + y^2 = 4 \qquad ...(ii)$$

Equation (i) is a circle with centre O at the origin and radius 2. Equation (ii) is a circle with centre C (2, 0) and radius 2.

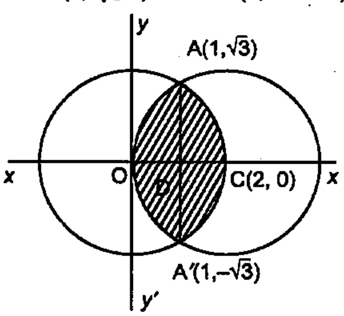
Solving equation (i) and (ii) we have

$$(x-2)^2 + y^2 = x^2 + y^2$$

or $\quad x^2 - 4x + 4 + y^2 = x^2 + y^2$

or $x = 1$ which gives $y = \pm\sqrt{3}$

Thus, the points of intersection of the given circles are A $(1, \sqrt{3})$ and A′$(1, -\sqrt{3})$



Required area of the enclosed region OACA′O between circles

= 2 [area of region ODCAO]

= 2 [area of region ODAO + area of region OCAD]

$$= 2\left[\int_0^1 y dx + \int_1^2 y dx\right]$$

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}dx + \int_1^2 \sqrt{4-x^2}dx\right] \text{ [from (i)]}$$

$$= 2\left[\frac{1}{2}(x-2)\sqrt{4-(x-2)^2} + \frac{1}{2}\times 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1$$
$$+ 2\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{1}{2}\times 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[(x-2)\sqrt{4-(x-2)^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_1^2$$
$$+ \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= \left[-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right) - 4\sin^{1}(-1)\right]$$
$$+ \left[4\sin^{-1}1 - \sqrt{3} - 4\sin^{-1}\frac{1}{2}\right]$$

$$= \left[\left(-\sqrt{3} - 4\times\frac{\pi}{6}\right) + 4\times\frac{\pi}{2}\right] + \left[4\times\frac{\pi}{2} - \sqrt{3} - 4\times\frac{\pi}{6}\right]$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \qquad \textbf{Ans.}$$

27. Find the vector and cartesian equations of the plane passing through the points having position vectors $\hat{i} + \hat{j} - 2\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$. Write the equation of a plane passing through a point (2, 3, 7) and parallel to the plane obtained above. Hence, find the distance between the two parallel planes. [6]

Solution : Let A, B,C be the points with position vectors $\hat{i} + \hat{j} - 2\hat{k}, 2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ respectively.

Then, $\quad \vec{AB}$ = P. V. of B – P. V. of A

$$= \left(2\hat{i} - \hat{j} + \hat{k}\right) - \left(\hat{i} + \hat{j} - 2\hat{k}\right)$$
$$= \hat{i} - 2\hat{j} + 3\hat{k}$$

and $\quad \vec{BC}$ = P. V. of C – P.V. of B

$$= \left(\hat{i} + 2\hat{j} + \hat{k}\right) - \left(2\hat{i} - \hat{j} + \hat{k}\right)$$
$$= -\hat{i} + 3\hat{j} + 0\hat{k}$$

A vector normal to the plane containing points A, B & C is

$$\vec{n} = \vec{AB} \times \vec{AC}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ -1 & 3 & 0 \end{vmatrix}$$

$$= -9\hat{i} - 3\hat{j} + \hat{k}$$

The required plane passes through the point having position vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and is normal to the vector $-9\hat{i} - 3\hat{j} + 2\hat{k}$. So, its vector equation is,

$$\left(\vec{r} - \vec{a}\right).\vec{n} = 0$$

$$\Rightarrow \quad \vec{r}.\vec{n} - \vec{a}.\vec{n} = 0$$

$$\Rightarrow \quad \vec{r}.\vec{n} = \vec{a}.\vec{n}$$

$$\Rightarrow \vec{r}.(-9\hat{i} - 3\hat{j} + \hat{k}) = (\hat{i} + \hat{j} + 2\hat{k}).\left(-9\hat{i} - 3\hat{j} + \hat{k}\right)$$

$$\Rightarrow \vec{r}.(-9\hat{i} - 3\hat{j} + \hat{k}) = -9 - 3 - 2$$

$$\Rightarrow \hat{r}.(-9\hat{i} - 3\hat{j} + \hat{k}) = -14$$

This is the required vector equation of the plane
The cartesian equation of plane is given by

$$(x\hat{i} + y\hat{j} + z\hat{k}).(-9\hat{i} - 3\hat{j} + \hat{k}) = -14$$

$$-9x - 3y + z = -14$$

$$9x + 3y - z = 14$$

Direction ratios of this plane are $(9, 3, -1)$

Then the equation of plane parallel to the above plane and passing through $(2,3,7)$ is

$$= a(x - x_1) + b(y - y_1) + c(z - z_1)$$

$$= 9(x - 2) + 3(y - 3) - 1(z - 7)$$

$$\Rightarrow \quad 9x + 3y - z - 20 = 0$$

This is the required parallel plane.

Then, Distance between $9x + 3y - z + 14 = 0$ and $9x + 3y - z - 20 = 0$

Let $P(x_1, y_1, z_1)$ be any point on $9x + 3y - z + 14 = 0$

Then,

$$9x_1 + 3y_1 - z_1 - 14 = 0$$

Let d be the distance between planes. Then,

d = length of perpendicular from $P(x_1, y_1, z_1)$ to $9x + 3y - z - 20 = 0$

$$d = \left|\frac{9x_1 + 3y_1 - z_1 - 20}{\sqrt{(9)^2 + (3)^2 + (-1)^2}}\right|$$

$$= \left|\frac{+14 - 20}{\sqrt{91}}\right|$$

$$= \frac{6}{\sqrt{91}} \text{ units.}$$ **Ans.**

OR

Find the equation of the line pasing through (2, –1, 2) and (5, 3, 4) and of the plane passing through (2, 0, 3), (1, 1, 5) and (3, 2, 4). Also, find their point of intersection.

Solution :

We know that the equation of line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

So equation of line is given by

$$\Rightarrow \quad \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$$

Now, equation of plane passing through points (2, 0,3), (1,1,5) and (3,2,4) is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 2 & y - 0 & z - 3 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x - 2)(-3) - y(-1 - 2) + (z - 3)(-2 - 1) = 0$$

$$\Rightarrow -3x + 6 + 3y - 3z + 9 = 0$$

$$\Rightarrow -x + y - z + 5 = 0 \quad \text{...(i)}$$

This is the required equation of plane.

Now,

Let $$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = k$$

$$\therefore \quad x = 3k + 2$$

$$y = 4k - 1$$

$$z = 2k + 2$$

On putting these values in equation (i), we have

$$-(3k + 2) + 4k - 1 - (2k + 2) + 5 = 0$$

$$\Rightarrow -3k - 2 + 4k - 1 - 2k - 2 + 5 = 0$$

$$\Rightarrow \quad -2k = 0$$

$$\Rightarrow \quad k = 0$$

Then, intersection points are

$$x = 3k + 2 = 2,$$

$$y = 4k - 1 = -1,$$

$$z = 2k + 2 = 2$$

$\therefore$ Point of interaction is $(2, -1, 2)$ **Ans.**

28. There are three coins. One is a two-headed coin, another is a biased coin that comes up heads 75% of the time and the third is an unbiased coin. One of the three coins is chosen at random and tossed. If it shows heads, what is the probability that it is the two-headed coin? [6]

Solution : Given, there are three coins.

Let, E_1 = coin is two headed

E_2 = biased coin

E_3 = unbiased coin

A = shows only head

Here, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Then $P\left(\frac{A}{E_1}\right) = 1$

$\left(\frac{A}{E_2}\right) = \frac{75}{100} = \frac{3}{4}$ (given)

$P\left(\frac{A}{E_3}\right) = \frac{1}{2}$

Now, Probability of two headed coin

$P(E_1/A)$

$$= \frac{P(E_1)P(A/E_1)}{P(E_1)\times P(A/E_1) + P(E_2)\times P(A/E_2) + P(E_3)\times P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times\frac{3}{4} + \frac{1}{3}\times\frac{1}{2}} = \frac{1}{\frac{4+3+2}{4}} = \frac{4}{9}$$ **Ans.**

29. A company produces two types of goods, A and B, that require gold and silver. Each unit of type A requries 3 g of silver and 1 g of gold while that of type B requires 1 g of silver and 2 g of gold. The company can use at the most of 9 g of silver and 8 of gold. If each unit of type A brings a profit of ₹ 40 and that of type B ₹ 50, find the number of units of each type that the company should produce to maximize profit. Formulate the above LPP and solve it graphically and also find the maximum profit [6]

Solution :

There are two types of goods, A and B and let units of type A be x and units of type B be y.

	A	B
Gold	1	2
Silver	3	1
Profit	40	50

Then, Total profit of goods

$P = 40x + 50y, x \geq 0, y \geq 0$

Hence, the mathematical formulation of the problem is as follows :

Maximise $P = 40x + 50y$ ...(i)

Subject to the constraints :

$x + 2y \leq 8$ ...(ii)

$3x + y \leq 9$ ...(iii)

$x \geq 0, y \geq 0$

To solve this LPP, we draw the lines

$x + 2y = 8$

$3x + y = 9$

$x = 0$ and $y = 0$

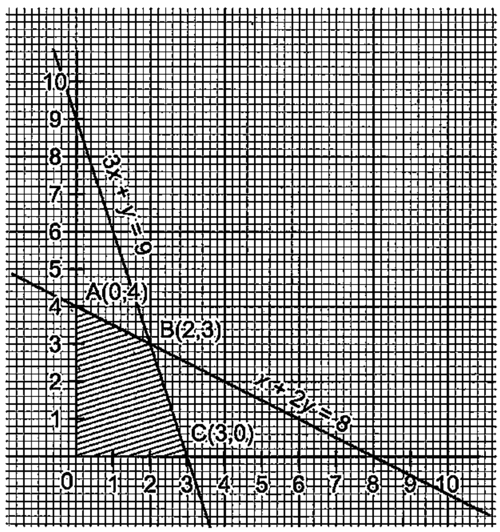
$x + 2y = 8$

x	0	8
y	4	0

and $3x + y = 9$

x	0	3
y	9	0

Plotting these points on the graph



The shaded region is the required feasible region.

Corner Points	Maximum $P = 40x + 50y$
A(0,4)	$0 + 50 \times 4 = 200$
B(2,3)	$2 \times 40 + 3 \times 50 = 230$
C(3,0)	$40 \times 3 + 0 = 120$
O (0,0)	$0 + 0 = 0$

Clearly, P is maximum at B, (2,3) and the maximum profit is ₹ 230. **Ans.**

Mathematics 2019 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

SECTION–A

1. Find $|AB|$, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$. [1]

Solution :

Given, $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

Then, $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore \quad |AB| = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ **Ans.**

2. Differentiate $e^{\sqrt{3x}}$, with respect to x. [1]

Solution :

Let $y = e^{\sqrt{3x}}$

On differentiating equation (i) w.r.t x, we get

$$\frac{dy}{dx} = e^{\sqrt{3x}} \times \sqrt{3} \times \frac{1}{2} x^{\frac{-1}{2}}$$

$$= \frac{\sqrt{3}}{2} \frac{e^{\sqrt{3x}}}{\sqrt{x}}$$ **Ans.**

SECTION–B

6. If $A = \begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$ and $|A^3| = 125$, the find the value of p. [2]

Solution :

Given, $A = \begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$ and $|A^3| = 125$

Now, $|A^3| = 125$

or $|A^3| = 5^3$

$|A| = 5$

Also we have $|A| = \begin{vmatrix} p & 2 \\ 2 & p \end{vmatrix}$

$\Rightarrow \quad p^2 - 4 = 5$

$\Rightarrow \quad p^2 = 9$

$\Rightarrow \quad p = \pm 3$ **Ans.**

12. Find the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ [2]

Solution :

Given, $\frac{dy}{dx} = e^{x+y}$

$\Rightarrow \quad \frac{dy}{dx} = e^x . e^y$

$\Rightarrow \quad \frac{dy}{e^y} = e^x . dx$

$\Rightarrow \quad e^{-y} \, dy = e^x . dx$

On integrating both sides, we get

$$\int e^{-y} dy = \int e^x dx$$

$\Rightarrow e^{-y} = e^x + c$, which is the required solution. **Ans.**

SECTION–C

21. If $(a + bx) e^{y/x} = x$, then prove that

$$x^3 \frac{d^2y}{dx^2} = \left(x \frac{dy}{dx} - y \right)^2$$ **[4]**

Solution : We have

$(a + bx) e^{y/x} = x$

$\Rightarrow \quad e^{y/x} = \frac{x}{a+bx}$

$\Rightarrow \quad \frac{y}{x} = \log \frac{x}{a+bx}$

$\Rightarrow \quad y = x \log \left(\frac{x}{a+bx} \right)$

$\Rightarrow \quad y = x \{ \log x - \log (a + bx) \}$

$\Rightarrow \quad \frac{y}{x} = \log x - \log (a + bx)$

On differentiating w.r.t. x, we get

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{1}{a+bx} \times b$$

$\Rightarrow \quad \frac{1}{x^2}\left(x \frac{dy}{dx} - y \right) = \frac{1}{x} - \frac{b}{a+bx}$

$\Rightarrow \quad x \frac{dy}{dx} - y = x^2 \left(\frac{1}{x} - \frac{b}{a+bx} \right)$

$$\Rightarrow \quad x\frac{dy}{dx} - y = \frac{ax}{a+bx} \quad \text{...(i)}$$

Again, differentiating both sides w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)a - ax\times b}{(a+bx)^2}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2 + abx - abx}{(a+bx)^2}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

On multiplying both sides by x^2, we get

$$x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2}$$

$$x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2 \quad \text{...(ii)}$$

From (i) and (ii), we get

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$

Hence Proved.

22. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of its edge is 12 cm? [4]

Solution :

Let x be the length of side, V be the volume and S be the surface area of cube.

Then, $V = x^3$ and $S = 6x^2$, where x is a function of time t

$$\text{Now,} \quad \frac{dV}{dt} = 8\,\text{cm}^3/\text{s (given)}$$

$$\therefore \quad 8 = \frac{dV}{dt} = \frac{d}{dt}(x^3) = 3x^2\frac{dx}{dt}$$

(By chain rule)

$$8 = 3x^2\frac{dx}{dt}$$

$$\frac{8}{3x^2} = \frac{dx}{dt} \quad \text{...(i)}$$

$$\text{Now,} \quad \frac{dS}{dt} = \frac{d(6x^2)}{dt}$$

$$\Rightarrow \quad \frac{dS}{dt} = 12x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dS}{dt} = 12\,x\,\frac{8}{3x^2} \quad \text{[Using (i)]}$$

$$= \frac{32}{x}$$

Hence, when $x = 12$ cm

$$\text{Then,} \quad \frac{dS}{dt} = \frac{32}{12} = \frac{8}{3}\ \text{cm}^2/\text{s}$$ **Ans.**

23. Find the cartesian and vector equations of the plane passing through the point A(2, 5, – 3), B(– 2, – 3, 5) and C(5, 3, – 3). [4]

Solution :

We know that the general equation of the plane passing through three points (x_1, y_1, z_1) (x_2, y_2, z_2), (x_3, y_3, z_3)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Then the plane passing through A (2, 5, –3), B (– 2, – 3, 5,), C (5, 3, +3)

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -2-2 & -3-5 & 5+3 \\ 5-2 & 3-5 & -3+3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(0+16) - (y-5)(0-24) + (z+3)(8+24) = 0$$

$$\Rightarrow \quad 16(x-2) + 24(y-5) + 32(z+3) = 0$$

$$\Rightarrow \quad 8[2x - 4 + 3y - 15 + 4z + 12] = 0$$

$$\Rightarrow \quad 2x + 3y + 4z - 7 = 0$$

$$\Rightarrow \quad 2x + 3y + 4z = 7$$

This is the required cartesian equation of the plane.

Now,

The required plane passes through the point A(2, 5, –3) whose position vector is $a = 2\hat{i} + 5\hat{j} - 3\hat{k}$ and is normal to the vector $\vec{n}$ given by $\vec{n} = \vec{AB} \times \vec{AC}$

$$\therefore \quad \vec{AB} = -2\hat{i} - 3\hat{j} + 5\hat{k} - (2\hat{i} + 5\hat{j} - 3\hat{k})$$

$$\vec{AB} = -4\hat{i} - 8\hat{j} + 8\hat{k} + 8\hat{k}$$

$$\vec{AC} = \left(5\hat{i} + 3\hat{j} - 3\hat{k} - \left(2\hat{i} + 5\hat{j} - 3\hat{k}\right)\right)$$

$$= 3\hat{i} - 2\hat{j}$$

Now, $\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}$

$$= 16\hat{i} - (-24)\hat{j} + (8+24)\hat{k}$$

$$= 16\hat{i} + 24\hat{j} + 32\hat{k}$$

The vector equation of the plane is given by

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k})$$

$$= (2\hat{i} + 5\hat{j} - 3\hat{k})(16\hat{i} + 24\hat{j} + 32\hat{k})$$

$$= 32 + 120 - 96$$

$$\Rightarrow \vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = 56$$

$$\Rightarrow \vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7$$

This is the required vector equation. **Ans.**

SECTION–D

24. Find the point on the curve $y^2 = 4x$, which is nearest to the point (2, – 8). [6]

Solution :

Given curve is of the form, $y^2 = 4x$ and let $p(x,y)$ is a point on the curve which is nearest to the point (2, – 8).

$\because \quad y^2 = 4x$

then $\quad \frac{y^2}{4} = x$

$\therefore$ P(x, y) will be $P\left(\frac{y^2}{4}, y\right)$

Now, the distance between point A and P is given by

$$AP = \sqrt{(x-2)^2 + (y+8)^2}$$

$$= \sqrt{\left(\frac{y^2}{4} - 2\right)^2 + (y+8)^2}$$

$$= \sqrt{\frac{y^4}{16} - y^2 + 4 + y^2 + 16y + 64}$$

$$= \sqrt{\frac{y^4}{16} + 16y + 68}$$

and let $Z = AP^2 = \frac{y^4}{16} + 16y + 68 \quad$ ...(i)

Now, differentiate equation (i) w.r.t. y, we get

$$\frac{dZ}{dy} = \frac{1}{16} \times 4y^3 + 16$$

$$= \frac{y^3}{4} + 16$$

For maximum or minumim value of Z, we have

$$\frac{dZ}{dy} = 0$$

$$\Rightarrow \frac{y^3}{4} + 16 = 0$$

$$\Rightarrow y^3 + 64 = 0$$

$$\Rightarrow (y+4)(y^2 - 4y + 16) = 0$$

$(\because y^2 - 4y + 16 = 0$ gives imaginary value of $y)$

$$\Rightarrow y = -4$$

Now, $\frac{d^2Z}{dy^2} = \frac{1}{4} \times 3y^2 = \frac{3}{4}y^2$

At $y = -4$,

$$\frac{d^2z}{dy^2} = \frac{3}{4}(-4)^2$$

$$= 12 > 0$$

Thus, Z is minimum when $y = -4$

Substituting $y = -4$ in equation of curve $y^2 = 4x$.

We have $x = 4$

Hence, the point (4, –4) on the curve $y^2 = 4x$ is nearest to the point (12, – 8). **Ans.**

25. Find $\int_1^3 (x^2 + 2 + e^{2x})\,dx$ as the limit of sums. [6]

Solution :

We have

$$\int_a^b f(x)dx = \lim_{h \to 0} h\,[f(a) + f(a+h) + f(a+2h) + ...f(a+(n-1)h]$$

where $h = \frac{b-a}{n}$

(i) Here, $a = 1$ and $b = 3$

$$f(x) = x^2 + 2 + e^{2x}$$

$$\therefore \quad h = \frac{3-1}{n} = \frac{2}{n} \Rightarrow nh = 2$$

Now, $I = \int_1^3 (x^2 + 2 + e^{2x})dx$

$$= \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) ....f(1+(n-1)h]$$

$$I = \lim_{h\to 0} h\ [(1+2+e^2) + \{(1+h)^2 + 2 + e^{2(1+h)}\} + \{(1+2h)^2 + 2 + e^{2(1+2h)} + ...\}\ \{(1+(n-1)h)^2 + 2 + e^{2(1+(n-1)h)}\}]$$

$$I = \lim_{h\to 0} h\left[1+2+e^2+1+h^2+2h + 2 + e^{2(1+h)} + 1 + 4h^2 + 4h + 2e^{2(1+2h)} + ...\{1+(n-1)^2h^2 + 2(n-1)h + 2 + e^{2(1+(n-1)h)}\}\right]$$

$$I = \lim_{h\to 0} h[1+1+1+......+2+2+.....2h+4h + 2(n-1)h + ... h^2 + 4h^2 + 4h^2 + (n-1)^2h^2 + ... + e^2 + e^{2(1+h)} + e^{2(1+2h)} + ...]$$

$$I = \lim_{h\to 0} h\left[1+1+1+......+2+2+.....2h+4h + 2(n-1)h + h^2 + 4h^2 + (n-1)^2h^2 + ... + e^2(1+e^{2h}+e^{4h}+.....e^{2(n-1)h}\right]$$

$$\left[\because 1+e^{2h}+e^{4h}+.....e^{2(n-1)h} = \frac{1(1-e^{2hn})}{1-e^{2n}}\right]$$

$$I = \lim_{h\to 0} h\left[n+2n+2n(1+2+3+.....n-1) + h^2(1+2^2+3^2+......(n-1)^2 + \frac{e^2\times(1-e^{2hn})}{1-e^{2n}}\right]$$

$$\left[\because \begin{matrix} 1+1+1+.....n = n \\ 2+2+2+n = 2n \end{matrix}\right]$$

$$I = \lim_{h\to 0} h\left[3n + 2h\frac{n(n-1)}{2} + \frac{h^2(n-1)n(2n-1)}{6} + e^2\frac{(1-e^{2hn})}{1-e^{2n}}\right]$$

$$\left[\because 1+2+3+4+.....n = \frac{n(n+1)}{2}\right.$$

and $$\left. 1^2+2^2+3^2+n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$I = \lim_{h\to 0}\left[3nh + nh^2(n-1) + \frac{h^3(n-1)n(2n-1)}{6} + \frac{e^2h(1-e^{2hn})}{1-e^{2n}}\right]$$

$$I = \lim_{h\to 0}\left[nh\times 3 + nh\times h(n-1) + \frac{nh\,h^2(n-1)(2n-1)}{6} + \frac{e^2h(1-e^{2hn})}{1-e^{2n}}\right]$$

Put $nh = 2$

$$I = \lim_{h\to 0}\left[6+2(2-h) + \frac{(nh-h)nh(2nh-h)}{6} + \frac{e^2}{2}\times\frac{2h(1-e^4)}{(1-e^{2n})}\right]$$

$$\lim_{h\to 0}\left[6+2+(2-h)+\frac{(2-h)\,2\,(4-h)}{6} + \frac{e^2}{2}(-1)(1-e^4)\right]$$

Now,

$$I = 6+4+\frac{2\times 2\times 4}{6} - \frac{e^2(1-e^4)}{2}$$

$$\Rightarrow \quad I = 10 + \frac{8}{3} - \frac{e^2}{2}(1-e^4)$$

$$\Rightarrow \quad I = \frac{38}{3} - \frac{e^2}{2}(1-e^4)$$ **Ans.**

OR

Using integration, find the area of the triangular region whose sides have the equation $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.

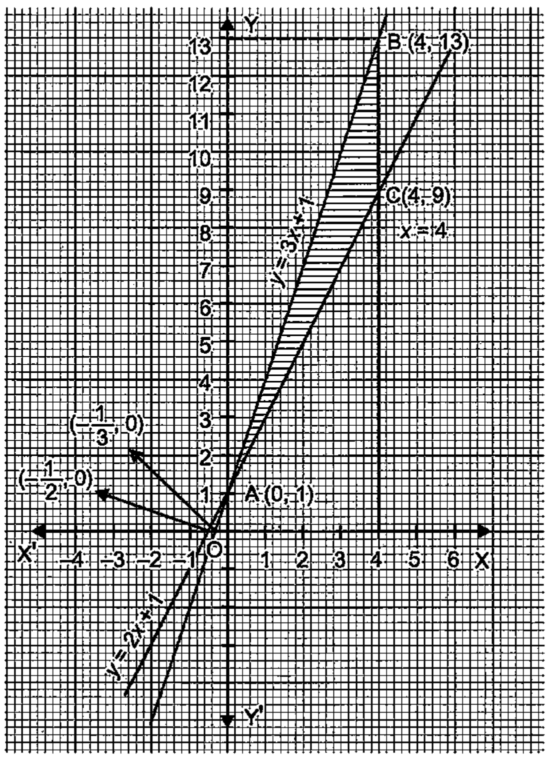
Solution :

The equations of sides of triangle are

$y = 2x + 1,$ ...(i)

$y = 3x + 1$ ...(ii)

and $x = 4$ ...(iii)



The equation $y = 2x + 1$ meets x and y axes at $\left(-\frac{1}{2}, 0\right)$ and (0,1). By joining these two points we obtain the graph of $x + 2y = 2$. Similarly, graphs of other equations are drawn.

Solving equation (i), (ii) and (iii) in pairs, we obtain the coordinates of verticies of ΔABC are A(0,1), B (4,13) and C (4,9).

Then, area of ΔABC = Area (OLBAO) – Area (OLCAO)

$$= \int_0^4 (3x+1)dx - \int_0^4 (2x+1)dx$$

$$= \int_0^4 (3x+1-2x-1)\,dx$$

$$= \int_0^4 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^4$$

$$= \frac{1}{2} \times 4 \times 4$$

= 8 square units. **Ans.**

Mathematics 2019 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

SECTION–A

1. Find the differenital equation representing the family of curves $y = ae^{2x} + 5$, where a is an arbitrary constant. [1]

Solution :

Given, $y = ae^{2x} + 5$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = a.e^{2x}.(2) + 0 \text{ [By Chain rule]}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2a.e^{2x} \quad \text{...(i)}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 2(a.e^{2x}.2)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\frac{dy}{dx} \quad \text{[using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$$

This is the required differential equation. **Ans.**

2. If $y = \cos\sqrt{3x}$, then find $\frac{dy}{dx}$. [1]

Solution :

Given, $y = \cos(\sqrt{3x})$

On differentiating w.r.t.x, we get

$$\frac{dy}{dx} = -\sin(\sqrt{3x}) \times \sqrt{3} \times \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{3}\sin(\sqrt{3x})}{2\sqrt{x}}$$ **Ans.**

SECTION–B

5. Show that the points $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ and $C(7\hat{i} - \hat{k})$ are collinear. [2]

Solution :

Given, $\vec{A} = -2\hat{i} + 3\hat{j} + 5\hat{k}$

$\vec{B} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{C} = 7\hat{i} + 0\hat{j} + \hat{k}$

Now, $\vec{AB}$ = position vector of B – position vector of A

$$= \hat{i} + 2\hat{j} + 3\hat{k} - (-2\hat{i} + 3\hat{j} + 5\hat{k})$$

$$= \hat{i} + 2\hat{j} + 3\hat{k} + 2\hat{i} - 3\hat{j} - 5\hat{k}$$

$$= 3\hat{i} - \hat{j} - 2\hat{k}$$

$\vec{BC}$ = position vector of C – position vector B

$$= 7\hat{i} + 0\hat{j} - 0\hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= 7\hat{i} + 0\hat{j} - \hat{k} - \hat{i} - 2\hat{j} - 3\hat{k}$$

$$= 6\hat{i} - 2\hat{j} - 4\hat{k}$$

$$= 2(3\hat{i} - \hat{j} - 2\hat{k})$$

Clearly, $\vec{BC} = 2\vec{AB} \Rightarrow AB \,||\, BC$

But B is a common point to $\vec{AB}$ and $\vec{BC}$.

$\therefore$ $\vec{AB}$ and $\vec{BC}$ are collinear vectors.

Hence, points A, B and C are collinear.

Hence Proved.

OR

Find , $|\vec{a} \times \vec{b}|$ if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$.

Solution :

Given, $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$

and $b = 3\hat{i} + 5\hat{j} - 2\hat{k}$

Then, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$

$$= \hat{i}(-2-15) - \hat{j}(-4-9) + \hat{k}(10-3)$$

$$= -17\hat{i} + 13\hat{j} + 7\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-17)^2 + (13)^2 + (7)^2}$$

$$= \sqrt{289 + 169 + 49} = \sqrt{507}$$ **Ans.**

6. Find : $\int \frac{x-5}{(x-3)^3} e^x dx$. **[2]**

Solution :

Let $I = \int \frac{x-5}{(x-3)^3} e^x dx$

$$I = \int \frac{(x-3)-2}{(x-3)^3} . e^x dx$$

$$= \int \frac{(x-3)-2}{(x-3)^3} . e^x dx$$

$$= \int \left[\frac{(x-3)}{(x-3)^3} - \frac{2}{(x-3)^3} \right] e^x dx$$

$$= \int \left[\frac{1}{(x-3)^2} - \frac{2}{(x-3)^3} \right] e^x dx \quad ...(i)$$

Now,

$$\frac{d}{dx}(x-3)^{-2} = -2(x-3)^{-3} = \frac{-2}{(x-3)^3}$$

and we know that

$$\int [f(x) + f'(x)]e^x dx = e^x f(x) + c$$

Here, $f(x) = \frac{1}{(x-3)^2}$ and $f'(x) = \frac{-2}{(x-3)^3}$

Then,

$$\int \frac{x-5}{(x-3)^3} e^x dx = e^x \times \frac{1}{(x-3)^2} + C = \frac{e^x}{(x-3)^2} + C$$

Ans.

SECTION–C

13. Solve for x :

$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$. **[4]**

Solution :

Given,

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$$

We know that

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$\therefore \tan^{-1}(x+1) + \tan^{-1}(x-1)$

$$= \tan^{-1}\left[\frac{x+1+x-1}{1-[(x+1)(x-1)]}\right]$$

$$= \tan^{-1}\left[\frac{2x}{1-(x^2-1)}\right]$$

$$= \tan^{-1}\left[\frac{2x}{2-x^2}\right]$$

Now, $\tan^{-1}\frac{2x}{2-x^2} = \tan^{-1}\frac{8}{31}$

$\Rightarrow \frac{2x}{2-x^2} = \frac{8}{31}$

$\Rightarrow 62x = 16 - 8x^2$

$\Rightarrow 31x = 8 - 4x^2$

$\Rightarrow 4x^2 + 31x - 8 = 0$

$\Rightarrow 4x^2 + 32x - x - 8 = 0$

$\Rightarrow 4x(x+8) - 1(x+8) = 0$

$\Rightarrow (x+8)(4x-1) = 0$

$\Rightarrow x = -8$ and $x = \frac{1}{4}$ **Ans.**

14. If $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, then prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$. **[4]**

Solution :

Given, $x = ae^t(\sin t + \cos t)$ ...(i)

and $y = ae^t(\sin t - \cos t)$ ...(ii)

On differentiating equation (i) with w.r.t. t, we get

$$\frac{dx}{dt} = a\frac{d}{dt}\left[e^t(\sin t + \cos t)\right] = a\frac{d}{dt}\left[e^t \sin t + e^t \cos t\right]$$

$$= a\left[e^t \cos t + \sin t\, e^t + e^t(-\sin t) + e^t \cos t)\right]$$

$$= a\left[e^t \cos t + e^t \sin t - \sin t + e^t + e^t \cos t\right]$$

$$= 2ae^t \cos t \quad ...(iii)$$

On differentiating equation (ii) w.r.t. t, we get

$$\frac{dy}{dt} = a\frac{d}{dt}\left[e^t(\sin t - \cos t)\right]$$

$$= a\frac{d}{dt}\left[e^t \sin t - e^t \cos t\right]$$

$$= a\left[e^t \cos t + \sin t\, e^t - \left(-e^t \sin t + \cos e^t\right)\right]$$

$$= a\left[e^t \cos t + e^t \sin t + \sin t\, e^t - e^t \cos t\right]$$

$$= 2ae^t \sin t \quad \text{...(iv)}$$

On dividing equation (iv) by (iii), we get

$$\frac{dy/dt}{dx/dt} = \frac{2ae^t \sin t}{2ae^t \cos t}$$

$$\frac{dy}{dx} = \frac{\sin t}{\cos t}$$

$$\text{L.H.S.} = \frac{dy}{dx} = \tan t \quad \text{...(v)}$$

Now, $\text{R.H.S.} = \frac{x+y}{x-y}$

$$= \frac{ae^t(\sin t + \cos t) + ae^t(\sin t - \cos t)}{ae^t(\sin t + \cos t) - ae^t(\sin t - \cos t)}$$

$$= \frac{2ae^t \sin t}{2ae^t \cos t} = \tan t \quad \text{...(vi)}$$

From equations (v) and (vi), we have

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$ **Hence Proved**

OR

Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x.

Solution :

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Let $x^{\sin x} = u$ ...(i)

and $(\sin x)^{\cos x} = v$ ...(ii)

Then,

$$y = u + v$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Now, $u = x^{\sin x}$

$$\log u = \sin x \log x$$

On differentiating w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \sin x \times \frac{1}{x} + \log x \cos x$$

$$\frac{du}{dx} = u\left(\frac{\sin x}{x} + \log x \cos x\right)$$

$$\frac{du}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right) \quad \text{...(iii)}$$

Now, $v = (\sin x)^{\cos x}$

$$\log v = \cos x \log \sin x$$

On differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \cos x \frac{1}{\sin x} \times \cos x + \log \sin x(-\sin x)$$

$$\frac{1}{v}\frac{dv}{dx} = \cot x \times \cos x + \sin x \log(\sin x)$$

$$\frac{dv}{dx} = v(\cot x \cos x - \sin x \log \sin x)$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}(\cot x - \cos x - \sin x \log \sin x) \quad \text{...(iv)}$$

So, $$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x \cos x\right) + (\sin x)^{\cos x}(\cot x \cos x - \sin x \log \sin x)$$

Ans.

15. Find :

$$\int \frac{2\cos x}{(1-\sin x)(2-\cos^2 x)}dx \quad [4]$$

Solution :

Let $$I = \int \frac{2\cos x\, dx}{(1-\sin x)(1-\sin^2 x)}$$

$$\Rightarrow \quad I = \int \frac{2\cos x}{(1-\sin x)(2-1+\sin^2 x)}dx$$

$$I = \int \frac{2\cos x\, dx}{(1-\sin x)(1-\sin^2 x)}$$

Now, let $\sin x = t$

$\Rightarrow$ $\cos x dx = dt$

Then, $$I = \int \frac{2dt}{(1-t)(1+t^2)}$$

Now, solving it by partial fraction,

$$\Rightarrow \quad \frac{2}{(1-t)(1+t^2)} = \frac{A}{(1-t)} + \frac{Bt+C}{(1+t^2)}.$$

$$\Rightarrow \quad 2 = A(1+t^2) + (Bt+C)(1-t)$$

$$\Rightarrow \quad 2 = A + At^2 + Bt - Bt^2 + C - Ct$$

$$\Rightarrow \quad 2 = t^2 + (A-B) + t(B-C) + A + C$$

Equating the coefficient of t^2, t and of constant terms of both sides, we get

$$A - B = 0$$

$$B - C = 0$$

and $$A + C = 2$$

On solving, we get

$$A = C = 1 \text{ and } B = 1$$

$$\therefore \int \frac{2dt}{(1-t)(1+t^2)} = \int\left[\frac{1}{(1-t)} + \left(\frac{t+1}{1+t^2}\right)\right]$$

$$I = \int \frac{1}{1-t}\,dt + \int \frac{t\,dt}{(1+t^2)} + \int \frac{1}{(1+t^2)}\,dt$$

$$= \log|1-t| + \frac{1}{2}\log|1+t^2| + \tan^{-1} t$$

$$= \log|1-\sin x| + \frac{1}{2}\log|1+\sin^2 x|$$

$$+ \tan^{-1}(\sin x) + c$$ **Ans.**

SECTION–D

24. Show that for the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, **[6]**

$A^3 - 6A^2 + 5A + 11\,I = 0$**. Hence, find** A^{-1}**.**

Solution :

Given, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

Now, $$A^3 = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8+28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7-9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

Now, putting A^3, A^2 in $A^3 - 6A^2 + 5A + 11\,I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$+ 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$+ \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24 & 7-12 & 1-6 \\ -23+18 & 27-48 & -69+84 \\ 32-42 & -13+18 & 58-84 \end{bmatrix}$$

$$+ \begin{bmatrix} 5+11 & 5+0 & 5+0 \\ 5+0 & 10-11 & -15+0 \\ 10-0 & -5+0 & 15-11 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & -5 & -5 \\ -5 & -21 & 15 \\ -10 & 5 & -26 \end{bmatrix} + \begin{bmatrix} 16 & 5 & 5 \\ 5 & 21 & -15 \\ 10 & -5 & 26 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$ **Hence Proved.**

We have, $A^3 - 6A^2 + 5A + 11I = 0$

Multiplying by A^{-1} on both sides, we get

$$A^{-1}(A^3 - 6A^2 + 5A + 11I) = 0.A^{-1}$$

$$\Rightarrow A^3A^{-1} - 6A^2.A^{-1} + 5A.A^{-1} + 11I.A^{-1} = 0$$

$$\Rightarrow A^2(AA^{-1}) - 6A.(AA^{-1}) + 5(AA^{-1}) + 11IA^{-1} = 0$$

$$\Rightarrow A^2I - 6AI + 5I + 11A^{-1} = 0$$

$$\left[\begin{array}{l} \because AA^{-1} = 1 \\ \text{and } 11IA^{-1} = 11A^{-1} \end{array}\right]$$

$$\Rightarrow A^2 - 6A + 5I + 11A^{-1} = 0$$

$$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I$$

$$\Rightarrow A^{-1} = \frac{1}{11}(-A^2 + 6A - 5I)$$

$$A^{-1} = \frac{-(A^2 - 6A + 5I)}{11}$$

$$A^{-1} = \frac{1}{11}\left(\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$A^{-1} = \frac{-1}{11}\left(\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)$$

$$A^{-1} = \frac{-1}{11}\left(\begin{bmatrix} 4-6 & 2-6 & 1-6 \\ -3-6 & 8-12 & -14+18 \\ 7+12 & -3+6 & 14+18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)$$

$$A^{-1} = \frac{-1}{11}\begin{bmatrix} -2 & -4 & -5 \\ -9 & -4 & 4 \\ -5 & 3 & -4 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{11}\begin{bmatrix} -2+5 & -4+0 & -5+0 \\ -9+0 & -4+5 & 4+0 \\ -5+0 & 3+0 & -4+5 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$ **Ans.**

OR

Using matrix method, solve the following system of equations :

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Solution :

The given equations are

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

These equations can be written in the from AX = B, where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

Now, $|A| = 3(2-3) + 2(4+4) + 3(-6-4)$

$= -17 \neq 0$

Hence, A is non-singular and so its inverse exists.

Confactors of A are

$A_{11} = -1,\ A_{12} = -8,\ A_{13} = -10$

$A_{21} = -5,\ A_{22} = -6,\ A_{23} = 1$

$A_{31} = -1,\ A_{32} = 9,\ A_{33} = 7$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}^T = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ 10 & 1 & 7 \end{bmatrix}$$

$$\text{Now,} \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B = \frac{-1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1, y = 2$ and $z = 3$ **Ans.**

26. A bag contains 5 red and 4 black balls, a second bag contains 3 red and 6 black balls. One of the two bags is selected at random and two balls are drawn at random (without replacement) both of which are found to be red. Find the probability that the balls are drawn from the second bag. [6]

Solution :

Let E_1 be the event of choosing the bag I, E_2 be the event of choosing the bag II and A be the event of drawing a red ball.

Then,

$$P(E_1) = P(E_2) = \frac{1}{2}$$

Now, $$P\left(\frac{A}{E_1}\right) = \frac{5}{9}\times\frac{4}{8} = \frac{20}{72}$$

and $$P\left(\frac{A}{E_2}\right) = \frac{3}{9}\times\frac{2}{8} = \frac{6}{72}$$

Now, the probability of drawing a ball from Bag II, if it is given that it is red is $P\left(\frac{E_2}{A}\right)$.

Now, by Bayes' theorem, we have

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2).P(A/E_2)}{P(E_1).P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_1}\right)}$$

$$= \frac{\frac{1}{2}\times\frac{6}{72}}{\frac{1}{2}\times\frac{20}{72}+\frac{1}{2}\times\frac{6}{72}}$$

$$= \frac{\frac{6}{2\times 72}}{\frac{20}{2\times 72}+\frac{6}{2\times 72}}$$

$$= \frac{6}{20+6}$$

$$= \frac{6}{26} = \frac{3}{13}$$ **Ans.**

Math 2019 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION–A

1. If A and B are square matrices of the same order 3, such that |A| = 2 and AB = 2I, write the value of |B|. **[1]**

Solution :

Given, $|A| = 2$ and $AB = 2I$

$\therefore$ $|A| = 2I$

$\Rightarrow$ $|A||B| = 2I$

$\Rightarrow$ $2|B| = 2I$

$\Rightarrow$ $|B| = I$ **Ans.**

2. If $f(x) = x + 1$, find $\frac{d}{dx}$ (fof) (x). **[1]**

Solution :

Given, $f(x) = x+1$

Now, $f^0 f(x) = f(f(x)$

$= f(x+1)$

$= x+1+1$

$= x+2$

$\therefore$ $\frac{d}{dx}(fof)(x) = 1$ **Ans.**

3. Find the order and the degree of the differential equation $x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$. **[1]**

Solution :

We have,

$$x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$$

$\therefore$ Order = 2 and degree = 1 **Ans.**

4. If a line makes angles 90°, 135°, 45° with x, y and z axes respectively, find its direction cosines. **[1]**

Solution :

Given, $\alpha = 90°$, $\beta = 135°$, $\gamma = 45°$

So, $l = \cos 90° = 0$

$m = \cos 135° = \cos(180° - 45°)$

$= -\cos 45° = -\frac{1}{\sqrt{2}}$

and $n = \cos 45° = \frac{1}{\sqrt{2}}$

$\therefore$ The required direction cosines are $0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$. **Ans.**

OR

Find the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the vector $2\hat{i}+2\hat{j}-3\hat{k}$. **[1]**

Solution :

Given, the line passes through the point (3, 4, 5) and parallel to the vector $2\hat{i}+2\hat{j}-3\hat{k}$.

D. R. s of the given vector are < 2, 2, – 3 >.

∴ Vector equation of line,

$$\vec{r}=(3\hat{i}+4\hat{j}+5\hat{k})+\lambda(2\hat{i}+2\hat{j}-3\hat{k})$$ **Ans.**

SECTION–B

5. Examine whether the operation* defined on R by $a*b = ab + 1$ is (i) a binary or not. (ii) if a binary operation, is it associative or not? [2]**

6. Find a matrix A such that 2A – 3B + 5C = 0, where $B=\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$. [2]

Solution :

Given, $B=\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}, C=\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$

and $2A-3B+5C = 0$

$\Rightarrow \quad 2A = 3B-5C$

$\Rightarrow \quad 2A = 3B-5C$

$\Rightarrow \quad A = \frac{1}{2}(3B-5C)$

$\Rightarrow \quad A = \frac{1}{2}\left(3\begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}\right)-5\begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$

$\Rightarrow \quad A = \frac{1}{2}\left(\begin{bmatrix}-6 & 6 & 0\\ 9 & 3 & 12\end{bmatrix}-\begin{bmatrix}10 & 0 & -10\\ 35 & 5 & 30\end{bmatrix}\right)$

$\Rightarrow \quad A = \frac{1}{2}\begin{bmatrix}-16 & 6 & 10\\ -26 & -2 & -18\end{bmatrix}$

$\Rightarrow \quad A = \begin{bmatrix}-8 & 3 & 5\\ -13 & -1 & -9\end{bmatrix}$ **Ans.**

7. Find : $\int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx$. [2]

Solution :

Let $\quad I = \int \frac{\sec^2 x\, dx}{\sqrt{\tan^2 x+4}}$

Let $\quad \tan x = t$

$\sec^2 x\, dx = dt$

∴ $\quad I = \int \frac{dt}{\sqrt{t^2+4}}$

$\Rightarrow \quad I = \int \frac{dt}{\sqrt{t^2+2^2}}$

$\Rightarrow \quad I = \log|(t-2)+\sqrt{t^2+4}|+C$

$\Rightarrow \quad I = \log|(\tan x+2)+\sqrt{\tan^2 x+4}|+C$ **Ans.**

8. Find : $\int\sqrt{1-\sin 2x}\,dx, \frac{\pi}{4}<x<\frac{\pi}{2}$. [2]

Solution :

Let $\quad I = \int\sqrt{1-\sin 2x}\,dx$

$I = \int\sqrt{(\cos x-\sin x)^2}\,dx$

$[\because 1-\sin 2A = (\cos A-\sin A)^2]$

$\Rightarrow \quad I = \int(\cos x-\sin x)dx$

$\Rightarrow \quad I = \sin x+\cos x+C$ **Ans.**

OR

Find : $\int \sin^{-1}(2x)\,dx$. [2]

Solution :

Let $\quad I = \int\sin^{-1}(2x)\,dx$

Let $\quad \sin^{-1}2x = t$

$2x = \sin t$

$2dx = \cos t\, dt$

∴ $\quad I = 2\int t\cos t\, dt$

$\Rightarrow \quad I = 2[t\sin t-\int 1\cdot\sin t\, dt]$

[Using by parts]

$\Rightarrow \quad I = 2[t\sin t+\cos t]+C$

$\Rightarrow \quad I = 2[2x\sin^{-1}2x+\cos(\sin^{-1}2x)]+C$

$\Rightarrow \quad I = 2[2x\sin^{-1}2x+\sqrt{1-4x^2}]+C$ **Ans.**

9. Form the differential equation representing the family of cures $y = e^{2x}(a + bx)$, where 'a' and 'b' are arbitrary constants. [2]

Solution :

Given, $\quad y = e^{2x}(a+bx)$ ...(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{2x}(b)+2(a+bx)e^{2x}$$

$$\frac{dy}{dx} = be^{2x}+2y \quad \text{[From (i)]}$$

$$\frac{dy}{dx}-2y = be^{2x} \quad ...(ii)$$

Again, differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx} = 2be^{2x}$$

** Answer is not given due to the change in present syllabus.

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx} = 2\left(\frac{dy}{dx}-2y\right) \quad \text{[From (ii)]}$$

$$\frac{d^2y}{dx^2}-\frac{2dy}{dx}=\frac{2dy}{dx}-4y$$

$$\frac{d^2y}{dx^2}-\frac{4dy}{dx}+4y=0$$

This is the required differential equation. **Ans.**

10. If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$. [2]

Solution :

Let $\vec{a}$ and $\vec{b}$ are two unit vectors.

Given, $|\vec{a}|=1, |\vec{b}|=1, |\vec{a}+\vec{b}|=1$

We know that,

$$|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2=2(|\vec{a}|^2+|\vec{b}|^2)$$

$$(1)^2+|\vec{a}-\vec{b}|^2 = 2(1^2+1^2)$$

$$\Rightarrow \quad (1)^2+|\vec{a}-\vec{b}|^2 = 4$$

$$\Rightarrow \quad |\vec{a}-\vec{b}|^2 = 3$$

$$\Rightarrow \quad |\vec{a}-\vec{b}|^2 = \sqrt{3}$$ **Ans.**

OR

If $\vec{a} = 2\hat{i}+3\hat{j}+\hat{k}$ + $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$ and $\vec{c}=-3\hat{i}+\hat{j}+2\hat{k}$, find $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$. [2]

Solution :

Given, $\vec{a}=2\hat{i}+3\hat{j}+\hat{k}$

$\vec{b}=\hat{i}+2\hat{j}+2\hat{k}$

$\vec{a}=-3\hat{i}+\hat{j}+2\hat{k}$

Now, $\left[\vec{a}, \vec{b}, \vec{c}\right] = \left[\vec{a}.(\vec{b}\times\vec{c})\right]$

$$= \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix}$$

$$= 2(-4-1)-3(2+3)+1(1-6)$$

$$= -10-15-5=-30$$ **Ans.**

11. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event "number is even" and B be the event "number is marked red". Find whether the events A and B are independent or not. [2]

Solution :

Given, S = { 1, 2, 3, 4, 5, 6 }

Let the two events be

A : The number is even

B : The number is red

$$P(A) = \frac{3}{6}=\frac{1}{2}, P(B)=\frac{3}{6}=\frac{1}{2}$$

and $$P(A\cap B) = \frac{1}{6}$$

$$\therefore \quad P(A\cap B) \neq P(A).P(B)$$

Hence, A and B are not independent **Ans.**

12. A die is thrown 6 times. If "getting an odd number" is a "success", what is the probability of (i) 5 successes? (ii) atmost 5 success? [2]**

OR

The random variable X has a probability distribution P(X) of the following form. where 'k' is some number

$$P(X=x) = \begin{cases} k, & \text{if } x=0 \\ 2k, & \text{if } x=1 \\ 3k, & \text{if } x=2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the value of 'k'. [2]

Solution :

We know that,

$$\Sigma P(X) = 1$$

$$P_{(x=0)}+P_{(x=1)}+P_{(x=2)}+P_{(x=\text{other})} = 1$$

$$\Rightarrow \quad K+2k+3k+0 = 1$$

$$\Rightarrow \quad 6K = 1$$

$$\Rightarrow \quad K = \frac{1}{6}$$ **Ans.**

SECTION–C

13. Show that the relation R on R defined as $R=[(a, b): a \leq b]$, is reflexive, and transitive but not symmetric. [2]

Solution :

Reflexive :

Let $a \in R$

$\therefore$ $a \leq a$

So, $(a, a) \in R$

Hence, R is reflexive.

** Answer is not given due to the change in present syllabus.

Symmetric :

Let $(a, b) \in R$

Then $(b, a) \in R$

Then, $a \leq b$

$\Rightarrow b \leq a$ (which is not true)

$\therefore (b, a) \notin R$

Hence, R is not symmetric.

Transitive :

Let, $a, b, c \in R$, such that $(a, b) \in R$ and $(b, c) \in R$

Then, $a \leq b$

and $b \leq c$

$\Rightarrow a \leq c$

$\Rightarrow (a, c) \in R$

Hence, R is transitive.

Hence, R is reflexive and transitive but not Symmetric. **Hence Proved.**

OR

Prove that the function $f : N \to N$, defined by $f(x) = x^2 + x + 1$ is one-one but not onto.
Find inverse of $f : N \to S$, where S is range of f.

Solution :

Given, $f : N \to N$, $f(x) = x^2 + x + 1$

Let A be the set of natural number (domain).

and B be the set of natural number (co-domain).

For One-One :

Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$

$\Rightarrow x_1^2 - x_2^2 + x_1 - x_2 = 0$

$\Rightarrow (x_1 - x_2)[x_1 + x_2 + 1] = 0$

$\Rightarrow x_1 - x_2 = 0$ $[\because x_1 - x_2 + 1 \neq 0]$

$\Rightarrow x_1 = x_2$

$\therefore$ f is one-one.

For Onto :

Let $y = 1 \in N$ (Co domain)

$\therefore x^2 + x 1 = 1$

$\Rightarrow x^2 + x = 0$

$\Rightarrow x(x + 1) = 0$

$\therefore x = 0, -1 \in N$

Not possible since domain = N

So, f is not onto.

Hence, f is one-one but not onto. **Hence Proved**

Now, $f : N \to S : f(x) = x^2 + x + 1$

where S = range (Given)

$f : N \to S$ is onto as co-domain = range.

Hence, f is invertible.

Let $y = f(x) \Rightarrow y = x^2 + x + 1$

$$\Rightarrow y = x^2 + 2x\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$$

$$\Rightarrow y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\Rightarrow y - \frac{3}{4} = \left(x + \frac{1}{2}\right)^2$$

$$\Rightarrow \frac{4y - 3}{4} = \left(x + \frac{1}{2}\right)^2$$

$$\Rightarrow \pm\frac{\sqrt{4y - 3}}{2} = x + \frac{1}{2} = \frac{2x + 1}{2}$$

$$\therefore x = \frac{-1 \pm \sqrt{4y - 3}}{2}$$

$$\Rightarrow x = \frac{-1 + \sqrt{4y - 3}}{2} \quad (\because X \in N)$$

Ans.

14. Solve : $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$. **[2]**

Solution : We have

$$\tan^{-1} 4x + \tan^{-1} 6x = \pi/4$$

$$\Rightarrow \tan^{-1}\left(\frac{4x + 6x}{1 - 4x \cdot 6x}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{10x}{1 - 24x^2} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{10x}{1 - 24x^2} = 1$$

$$\Rightarrow 10x = 1 - 24x^2$$

$$\Rightarrow 24x^2 + 10x - 1 = 0$$

$$\Rightarrow x = \frac{-10 \pm \sqrt{100 + 96}}{48}$$

$$\Rightarrow x = \frac{-10 \pm 14}{48}$$

$$\Rightarrow x = \frac{-1}{2}, \frac{1}{12}$$

$$\therefore x = \frac{1}{12}$$

Ans.

15. Using properties of determinants, prove that

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3.$$ **[2]**

Solution :

$$\text{L.H.S} = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and we get $R_3 \to R_3 - R_1$, we get

$$= \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 1-a^2 & -a+1 & 0 \\ 3-a^2-2a & -2a+2 & 0 \end{vmatrix}$$

Taking $(1-a)$ common from R_2 & R_3, we get

$$= (1-a)^2 \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ (1-a) & 1 & 0 \\ a-3 & 2 & 0 \end{vmatrix}$$

Expanding along C_3, we get

$$= (1-a)^2[1(2+2a)-(a+3)]$$
$$= (1-a)^2[2+2a+a-3]$$
$$= (1-a)^2(a-1)$$
$$= (a-1)^2(a-1)$$
$$= (a-1)^3 = \text{R.H.S.}$$

Hence Proved.

16. If $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, show that

$$\frac{dy}{dx} = \frac{x+y}{x-y}.$$ **[2]**

Solution :

Given, $\log\left(x^2+y^2\right) = 2\tan^{1}\left(\frac{y}{x}\right)$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{x^2+y^2}\left(2x+2y\frac{dy}{dx}\right) = \frac{2}{1+\frac{y^2}{x^2}}\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)$$

$$\Rightarrow \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2}\cdot\frac{dy}{dx} = \frac{1}{x^2+y^2}\left(x\frac{dy}{dx}-y\right)$$

$$\Rightarrow \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2}\cdot\frac{dy}{dx} = \frac{x}{x^2+y^2}\cdot\frac{dy}{dx} - \frac{y}{x^2+y^2}$$

$$\Rightarrow \frac{x}{x^2+y^2} + \frac{y}{x^2+y^2} = \frac{x}{x^2+y^2}\cdot\frac{dy}{dx} - \frac{y}{x^2+y^2}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{x+y}{x^2+y^2} = \left(\frac{x-y}{x^2+y^2}\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}.$$ **Ans.**

OR

If $x^y - y^x = a^b$, find $\frac{dy}{dx}$

Solution :

$$x^y + y^x = a^b$$

Let $u = x^y$ and $v = y^x$

Then, $u - v = a^b$

$$\frac{du}{dx} - \frac{dv}{dx} = 0 \quad \text{...(i)}$$

Now, $u = x^y$

Taking log on both sides, we get

$$\log u = x \log x$$

Differentiating both sides w.r.t. x, we get.

$$\frac{1}{u}\cdot\frac{du}{dx} = \frac{y}{x} + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right]$$

and $v = y^x$

Taking log on both sides, we get

$$\log v = x \log y$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{x}{y}\cdot\frac{dy}{dx} + \log y$$

$$\Rightarrow \frac{dv}{dx} = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right]$$

From equation (i),

$$x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right] - y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] = 0$$

$$\Rightarrow y.x^{y-1} + x^y \log x.\frac{dy}{dx} - xy^{x-1}.\frac{dy}{dx} - y^x \log y = 0$$

$$\Rightarrow \frac{dy}{dx}\left[x^y \log x - xy^{x-1}\right] = y^x \log y - yx^{y-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \left[\frac{y^x \log y - y.xy^{-1}}{x^y \log x - xy^{n-1}}\right]$$

Ans.

17. If $y = (\sin^{-1} x)^2$, prove that

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0 \qquad [2]$$

Solution :

Given, $y = (\sin^{-1}x)^2$ ...(i)

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{2\sin^1 x}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = 2\sin^{-1} x \qquad ...(ii)$$

Again differentiating both sides w.r.t. x, we get

$$\sqrt{1-x^2}\frac{d^2y}{dx^2} = \frac{2x}{2\sqrt{1-x^2}}\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}\frac{dy}{dx} - x\frac{dy}{dx} - 2 = 0$$

Hence Proved.

18. Find the equation of tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$. Also, write the equation of normal to the curve at the point of contact. [2]

Solution :

Given, $y = \sqrt{3x-2}$ ...(i)

Differentiating both sides w.r.t.x, we get

$$\frac{dy}{dx} = \frac{2}{2\sqrt{3x-2}} = m_1$$

and equation of line, $4x - 2y + 5 = 0$

$$\text{Slope} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } y} = \frac{-4}{-2} = 2 = m_2$$

Apply condition of parallel line,

$$\frac{3}{2\sqrt{3x-2}} = 2$$

$$\Rightarrow \quad 3 = 4\sqrt{3x-2}$$

$$\Rightarrow \quad 9 = 16(3x-2)$$

$$\Rightarrow \quad 9 = 48x - 32$$

$$41 = 48x$$

$$\Rightarrow \quad x = \frac{41}{48}$$

Putting the value of x in (i), we get

$$y = \sqrt{3\left(\frac{41}{48}\right) - 2} = \frac{3}{4}$$

$\therefore$ Point of contact $\left(\frac{41}{48}, \frac{3}{4}\right)$

Slope of curve = 2

So, equation of Tangent is

$$\left(y - \frac{3}{4}\right) = 2\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \quad \left(\frac{4y-3}{4}\right) = 2\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow \quad \frac{4y-3}{4} = \frac{48x-41}{28}$$

$$\Rightarrow \quad \frac{4y-3}{4} = \frac{48x-41}{24}$$

$$\Rightarrow \quad 4y - 3 = \frac{48x-41}{6}$$

$$\Rightarrow \quad 24y - 18 = 48x - 41$$

$$\Rightarrow \quad 48x - 24y - 23 = 0$$

Ans.

And equation of normal is

$$\left(y - \frac{3}{4}\right) = \frac{-1}{2}\left(x - \frac{41}{48}\right)$$

$$\Rightarrow \quad 4y - 3 = \frac{-1}{2}\left(\frac{48x-41}{48}\right)$$

$$\Rightarrow \quad 96(4y-3) = -48x + 41$$

$$\Rightarrow \quad 384y - 288 = -48x + 41$$

$$\Rightarrow \quad 48x + 384y - 329 = 0$$

Ans.

19. Find : $\int \frac{3x+5}{x^2+3x-18}dx.$ [2]

Solution :

Let $\quad I = \int \frac{3x+5}{x^2+3x-18}dx$

$$(3x+5) = A(2x+3) + B$$

Comparing coefficients of x,

$$3 = 2A$$

$$\therefore \quad \frac{3}{2} = A$$

Comparing contant terms,

$$5 = 3A + B$$

$$5 = \frac{9}{2} + B$$

$$B = \frac{1}{2}$$

$$\therefore \quad I = \frac{3}{2}\int \frac{2x+3}{x^2+3x-18}dx + \frac{1}{2}\int \frac{dx}{x^2+3x-18}$$

Let $I_1 = \frac{3}{2}\int \frac{2x+3}{x^2+3x-18}dx$

Put $x^2 + 3x - 18 = t$

$(2x+3)\,dx = dt$

$$I_1 = \frac{3}{2}\int \frac{dt}{t}$$

$$\Rightarrow \quad I_1 = \frac{3}{2}\log|t| + C_1$$

$$\Rightarrow \quad I_1 = \frac{3}{2}\log|x^2+3x-18| + C_1$$

Now, $I_2 = \frac{1}{2}\int \frac{dx}{x^2+3x-18}$

$$\Rightarrow \quad I_2 = \frac{1}{2}\int \frac{dx}{x^2 + 2x.\frac{3}{2} + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 18}$$

$$\Rightarrow \quad I_2 = \frac{1}{2}\left(\frac{dx}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right)$$

$$\Rightarrow \quad I_2 = \frac{1}{2} \times \frac{2}{2\times 9}\log\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right| + C_2$$

$$\Rightarrow \quad I_2 = \frac{1}{18}\log\left|\frac{2x-6}{2x+12}\right| + C_2$$

$$\Rightarrow \quad I_2 = \frac{1}{18}\log\left|\frac{x-3}{x+6}\right| + C_2$$

$$\therefore \quad I = I_1 + I_2$$

$$I = \frac{3}{2}\log\left|x^2+3x-18\right| + \frac{1}{18}\log\left|\frac{x-3}{x+6}\right| + C$$

Ans.

20. Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, hence evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx.$ **[2]**

Solution :

To prove : $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof : Let $x = a - t$

$dx = -dt$

Also, $x = 0 \Rightarrow t = a$

and $x = a \Rightarrow t = 0$

$$\therefore \quad \int_0^a f(x)dx = -\int_a^0 f(a-t)dt$$

$$\Rightarrow \quad \int_0^a f(x)dx = \int_0^a f(a-t)dt$$

$$\left[\int_b^a f(x)dx = -\int_b^a f(x)dx\right]$$

$$\Rightarrow \quad \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$$

Hence Proved.

Let $I = \int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x}$ ...(i)

$$I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$$

[Applying above property]

$I = \int_0^{\pi} \frac{(\pi-x)\sin dx}{1+\cos^2 x}$ ...(ii)

$$\left[\begin{array}{l}\because \sin(\pi-\theta) = \sin\theta \\ \cos(\pi-\theta) = \cos\theta\end{array}\right]$$

Adding equation (i) and (ii), we get

$$2I = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Put $\cos x = t$

$\sin x\,dx = dt$

when $x = 0, t = 1$

and $x = \pi, t = -1$

$$\Rightarrow \quad 2I = -\pi\int_1^{-1} \frac{dt}{1+t^2}$$

$$\Rightarrow \quad 2I = -\pi\left[\tan^{-1}(t)\right]_1^{-1}$$

$$\Rightarrow \quad 2I = -\pi\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$\Rightarrow \quad 2I = -\pi[-\tan^{-1}(1) - \tan^{-1}(1)]$$

$$\left[\because \tan^{-1}(-x) = \tan^{-1} x\right]$$

$$\Rightarrow \quad 2I = -\pi\left[-2\tan^{-1}(1)\right]$$

$$\Rightarrow \quad 2I = -\pi\left(-\frac{2\pi}{4}\right)$$

$$\Rightarrow \quad 2I = \frac{2\pi^2}{4}$$

$$\Rightarrow \quad I = \frac{\pi^2}{4}$$

Ans.

21. Solve the differential equation $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$, given that $y = 0$ when $x = 1$. [2]

Solution :

Given, $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$

$$\Rightarrow \quad x\,dy = (\sqrt{x^2+y^2} + y)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{x}$$

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So, $$v + x.\frac{dv}{dx} = \frac{\sqrt{x^2+v^2x^2}+vx}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$$

$$\Rightarrow \quad x.\frac{xv}{dx} = \sqrt{1+v^2}$$

Integrating both sides, we get

$$\Rightarrow \quad \int\frac{dv}{\sqrt{1+v^2}} = \int\frac{dx}{x}$$

$$\log\left|v + \sqrt{1+v^2}\right| = \log x + \log c$$

$$\log\left|v + \sqrt{1+v^2}\right| = \log(xc)$$

$$v + \sqrt{1+x^2} = xc$$

$$\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = x\int c$$

$$\frac{y+\sqrt{x^2+y^2}}{x} = xc$$

$$y + \sqrt{x^2+y^2} = x^2c$$

Now, Given $y = 0$ when $x = 1$

$\therefore$ From equation (i),

$$0 + 1 = c$$

$$c = 1$$

Substitute the value of c in (i), we get

$$y + \sqrt{x^2+y^2} = x^2$$

Ans.

OR

Solve the differential equation $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$, subject to the initial condition $y(0) = 0$. [2]

Solution :

Given, $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2xy}{1+x^2} - \frac{4x^2}{1+x^2} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{4x^2}{1+x^2}$$

Here, $P = \frac{2xy}{1+x^2}$ and $Q = \frac{4x^2}{1+x^2}$

$$\text{I.F} = e^{\int pdx} = e^{\int\frac{2x}{1+x^2}dx}$$

$$\text{I.F.} = e^{\log|1+x^2|}$$

$$\text{I.F.} = 1 + x^2 \qquad \left[\because e^{\log x} = x\right]$$

Solution of given differential equation is

$$y\,(\text{I.F.}) = \int(\text{I.F} \times Q)\,dx$$

$$\Rightarrow \quad y\,(1+x^2) = \int(1+x^2)\left(\frac{4x^2}{1+x^2}\right)dx$$

$$\Rightarrow \quad y\,(1+x^2) = \frac{4x^3}{3} + C \qquad ...(i)$$

Now, Puting $x = 0, y = 0$ in (i), we get

$$0 = 0 + C$$

$$\therefore \quad C = 0$$

Substitute the value of C in (i), we get

$$y(1+x^2) = \frac{4x^3}{3}$$

$$y = \frac{4x^3}{3(1+x^2)} \qquad \textbf{Ans.}$$

22. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\vec{AB}$ and $\vec{CD}$ are collinear or not. [2]

Solution :

$A(\hat{i}+\hat{j}+\hat{k})$, $B(2\hat{i}+5\hat{j})$, $C(3\hat{i}+2\hat{j}-3\hat{k})$ and

$D(\hat{i}-6\hat{j}-\hat{k})$.

Now, $\vec{AB} = \hat{i}+4\hat{j}-\hat{k}$

$\vec{CD} = -2\hat{i}-8\hat{j}+2\hat{k}$

Let θ be the angle between $\vec{AB}$ & $\vec{CD}$

So, $$\cos\theta = \frac{\vec{AB}.\vec{CD}}{|\vec{AB}||\vec{CD}|}$$

$$\cos\theta = \frac{(\hat{i}+4\hat{j}-\hat{k}).(-2\hat{i}-8\hat{j}+2\hat{k})}{\sqrt{1+16+1}\sqrt{4+64+4}}$$

$$\cos\theta = \frac{-2-32-2}{\sqrt{18}\sqrt{72}}$$

$$\cos\theta = \frac{-36}{\sqrt{2}.6\sqrt{2}}$$

$$\cos\theta = -1$$

$$\cos\theta = \cos\pi$$

$$\theta = \pi \qquad \textbf{Ans.}$$

23. Find the value of λ so that the line $\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not. [2]

Solution :

Equation of 1st line,

$$\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$$

$$\Rightarrow \quad \frac{x-1}{-3}=\frac{y-2}{\lambda/7}=\frac{z-3}{2}$$

DR's of 1st line $<-3, \lambda/7, 2>$

and equation of 2nd line,

$$\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$$

$$\Rightarrow \quad \frac{x-1}{-3\lambda/7}=\frac{y-5}{1}=\frac{6-z}{-5}$$

DR's of 2nd line $\left\langle\frac{-3\lambda}{7}, 1, -5\right\rangle$

Since, given lines are at right angles,

$$-3\left(\frac{-3\lambda}{7}\right)+\frac{\lambda}{7}(1)+2(-5)=0$$

$$\Rightarrow \quad \frac{9\lambda}{7}+\frac{\lambda}{7}-10=0$$

$$\Rightarrow \quad 10\lambda-70=0$$

$$\Rightarrow \quad 10\lambda=70$$

$$\Rightarrow \quad \lambda=7$$

Now, equation of 1st line,

$$\frac{x-1}{-3}=\frac{y-2}{1}=\frac{z-3}{2}$$

and equation of 2nd line,

$$\frac{x-1}{-3}=\frac{y-5}{1}=\frac{z-6}{-5}$$

Here, $\vec{a_1} = \hat{i}+2\hat{j}+3\hat{k}$, $\vec{b_1} = -3\hat{i}+\hat{j}+2\hat{k}$

$\vec{a_2} = \hat{i}+5\hat{j}+6\hat{k}$, $\vec{b_2} = -3\hat{i}+\hat{j}=5\hat{k}$

$$\therefore \quad S.D = \left|\frac{(\vec{a_2}-\vec{a_1}).(\vec{b_1}\times\vec{b2})}{|\vec{b_1}\times\vec{b2}|}\right|$$

$$(\vec{a_2}-\vec{a_i}) = 0\hat{i}+3\hat{j}+3\hat{k}$$

and $$(\vec{b_1}\times\vec{b_2}) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -3 & 1 & 2\\ -3 & 1 & -5\end{vmatrix}$$

$$= \hat{i}(-5-2)-\hat{j}(15+6)+\hat{k}(-3+3)$$

$$= -7\hat{i}-11\hat{j}$$

$$\therefore \quad S.D. = \frac{\left|3\hat{i}+3\hat{k}-(-7\hat{k}-11\hat{j})\right|}{\sqrt{49+121}}$$

$$S.D. = \left|\frac{-21-33}{\sqrt{170}}\right|$$

$$\text{S.D.} = \left|\frac{-54}{\sqrt{170}}\right| = \frac{54}{\sqrt{170}} \neq 0$$

Hence, The given two lines do not intersect each other. **Ans.**

SECTION-D

24. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, find A^{-1}. Hence, solve the system of equations $x + y + z = 6, x + 2z = 7,$ $3x + y + z = 12$ **[6]**

Solution :

Given,

$$A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix}$$

$$|A| = 1(0-2) - 1\,(1-6) + 1(1-0)$$

$$|A| = -2 + 5 + 1$$

$$|A| = 4 \neq 0$$

$\therefore A^{-1}$ exits.

Now, cofactors of A are,

$a_{11} = 2,\quad a_{21} = 0,\quad a_{31} = 2$

$a_{12} = 5,\quad a_{22} = -2,\quad a_{32} = -1$

$a_{13} = 1,\quad a_{23} = 2,\quad a_{33} = -1$

$$\text{adj } A = \begin{vmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{vmatrix}^T$$

$$\text{adj } A = \begin{vmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{vmatrix}$$

So, $$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \quad \text{...(i) Ans.}$$

The given equations are :

$$x + y + z = 6,$$
$$x + 2z = 7,$$
and $$3x + y + z = 12$$

Here, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix} \quad \text{[Using (i)]}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -12+0+24 \\ 30-14+12 \\ 6+14-12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore \quad x = 3, \quad y = 1 \quad$ and $\quad z = 2$ **Ans.**

OR

Find the inverse of the following matrix using elementary operations.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution :

Given, $$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$A = IA$$

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_2 \to R_2 + R_1$, we get

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Now, Applying $R_1 \to R_1 + R_3$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_2 \to \frac{1}{5}R_2$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{2}{5} \\ 0 & -2 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_3 \to R_3 + 2R_2$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{2}{5} \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{2}{5} & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + 2R_3$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ \frac{2}{5} & \frac{1}{5} & 1 \end{bmatrix} A$$

Now, Applying $R_3 \to 5R_3$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A$$

Hence, $$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$ **Ans.**

25. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m³. If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square metre for the sides, what is the cost of least expresive tank? [6]

Solution :

Let the length and breadth of the tank be x and y metres, respectively.

Given, Volume = 8 m³

[∵ depth of tank = 2 m]

$$\Rightarrow \quad 2xy = 8$$

$$\Rightarrow \quad y = \frac{8}{2x} = \frac{4}{x}$$

Let C be the cost of tank. Then

$$C = 70\,xy + 45\,(2 \times 2x + 2 \times 2y)$$

$$\Rightarrow \quad C = 70\,xy + 180x + 180\,y$$

$$\Rightarrow \quad C = 70\,x \times \frac{4}{x} + 180\,x + 180 \times \frac{4}{x}$$

$$\Rightarrow \quad C = 280 + 180x + \frac{720}{x} \qquad \text{(i)}$$

Now, differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = 180 - \frac{720}{x^2}$$

For maxima and minima, $\frac{dC}{dx} = 0$

$$180 - \frac{720}{x^2} = 0$$

$$\Rightarrow \quad 180 = \frac{720}{x^2}$$

$$\Rightarrow \quad x^2 = 4$$

$$\Rightarrow \quad x = \pm 2$$

$$\Rightarrow \quad x = 2$$

Again, Differentiating both sides w.r.t. x, we get

$$\frac{d^2C}{dx^2} = \frac{1440}{x^3}$$

$$\left.\frac{d^2C}{dx^2}\right|_{x=2} = \frac{1440}{8} = 180 > 0$$

∴ when $x = 2$, cost of tank is minimum

Substituting the value of x in equation (i), we get

$$C = 280 + 180 \times 2 + \frac{720}{2}$$

$$\Rightarrow \quad C = 280 + 360 + 360$$

$$\Rightarrow \quad C = 1000$$

Hence, the cost of least expensive tank is ₹ 1000. **Ans.**

26. Using integration, find the area of a triangle ABC, whose vertices are A(2, 5), B(4, 7) and C(6, 2). [6]

Solution :

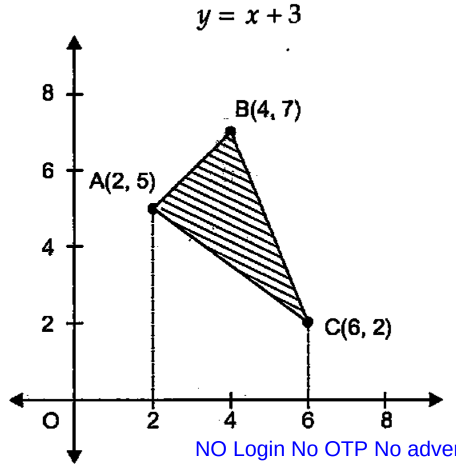
Given, A (2,5), B (4,7) and C (6,2) be the vertices of a triangle.

The equation of side AB.

$$(y - 5) = \frac{7-5}{4-2}(x - 2)$$

$$\Rightarrow \quad y - 5 = x - 2$$

$$\Rightarrow \quad y = x + 3$$



The equation of side BC,

$$(y-7) = \frac{2-7}{6-4}(x-4)$$

$$\Rightarrow \quad y-7 = \frac{-5}{2}(x-4)$$

$$\Rightarrow \quad 2y-14 = -5x+20$$

$$\Rightarrow \quad 2y = -5x+34$$

$$\Rightarrow \quad y = \frac{-1}{2}(5x-34)$$

The equation of sides AC,

$$(y-5) = \frac{2-5}{6-2}(x-2)$$

$$\Rightarrow \quad y-5 = \frac{-3}{4}(x-2)$$

$$\Rightarrow \quad 4y-20 = -3x+6$$

$$\Rightarrow \quad 4y = -3x+26$$

$$y = -\frac{1}{4}(3x-26)$$

∴ Area of ΔABC

$$= \int_2^4 y_{AB}dx + \int_4^6 y_{BC}dx - \int_2^6 y_{AC}dx$$

$$= \int_2^4 (x+3)\,dx + \int_4^6 -\frac{1}{2}(5x-34)\,dx - \int_2^6 -\frac{1}{4}(3x-26)\,dx$$

$$= \left[\frac{x^2}{2}+3x\right]_2^4 - \frac{1}{2}\left[\frac{5x^2}{2}-34x\right]_4^6 + \frac{1}{4}\left[\frac{3x^2}{2}-26x\right]_2^6$$

$$= \left[\left(\frac{16}{2}+12\right)-\left(\frac{4}{2}+6\right)\right] - \frac{1}{2}\left[\left(\frac{180}{2}-204\right) - \left(\frac{80}{2}-136\right)\right] + \frac{1}{4}\left[\left(\frac{108}{2}-156\right)-\left(\frac{12}{2}-52\right)\right]$$

$$= [(8+12)-(2+6)] - \frac{1}{2}[(90-204)-(40-136)] + \frac{1}{4}[(54-156)-(6-52)]$$

$$= 12+\frac{1}{2}(18)-\frac{1}{4}(56) = 12+9-14 = 7 \text{ sq. units.}$$ **Ans.**

OR

Find the area of a region lying above x-axis and included between the circle $x^2+y^2=8x$ and inside of the parabola $y^2=4x$.

Solution :

Given, equation of circle is $x^2+y^2=8x$ can be expressed as

$$(x-4)^2+y^2 = 16 \qquad \text{...(i)}$$

Centre is (4,0) and radius is 4

and equation of parabola is $y^2=4x$ ...(ii)

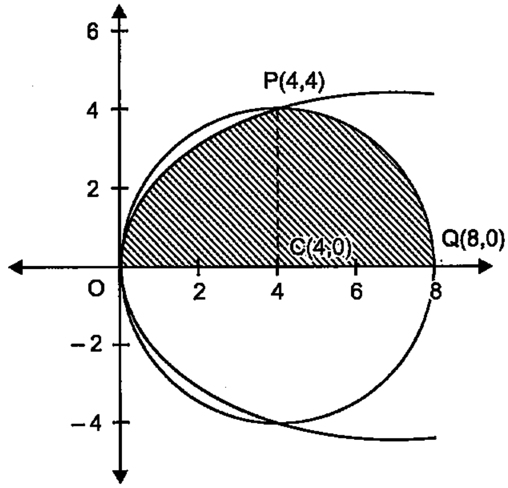
Let Required Area = I

$$\therefore \text{I} = \int_0^4 y \text{ of the parabola } dx = \int_4^8 y \text{ of the circle } dx$$

$$= \underbrace{\int_0^4 2\sqrt{x}\,dx}_{I_1} + \underbrace{\int_4^8 \sqrt{4^2-(x-4)^2}\,dx}_{I_2}$$

$$I_1 = 2\int_0^4 \sqrt{x}\,dx$$

$$I_1 = 2\left[\frac{x^{3/2}}{3/2}\right]_0^4$$



$$I_1 = \frac{4}{3}[4^{\frac{3}{2}}-0]$$

$$I_1 = \frac{4}{3}(8) = \frac{32}{3}$$

and $$I_2 = \int_4^8 \sqrt{4^2-(x-4)^2}\,dx$$

$$I_2 = \left[\frac{x-4}{2}\sqrt{4^2-(x-4)^2} + \frac{16}{2}\sin^{-1}\left(\frac{x-4}{4}\right)\right]_4^8$$

$$I_2 = \frac{1}{2}\left[\left(\frac{4}{2}(0)+16\sin^{-1}(1)\right)-(0)\right]$$

$$I_2 = \frac{1}{2}\times 16\frac{\pi}{2} = 4\pi$$

$$\therefore \quad I = I_1 + I_2$$

$$= \left(\frac{32}{3}+4\pi\right)$$

$$= \frac{4}{3}(8+3\pi) \text{ sq. units.}$$ **Ans.**

27. Find the vector and Cartesian equation of the plane passing through the points (2, 2, –1), (3, 4, 2) and (7, 0, 6). Also find the vector equation of a plane passing through (4, 3, 1) and parallel to the plane obtained above. **[6]**

Solution :

Let A (2, 2, – 1), B (3, 4, 2,) and C (7, 0, 6)

The equation of plane passing through A(2, 2, – 1),

$$a(x-2)+b(y-2)+c(z+1) = 0 \quad ...(i)$$

Since (3, 4, 2) and (7,0,6) lies on plane

$$\therefore \quad a+2b+3c = 0 \quad ...(ii)$$

$$\text{and} \quad 5a-2b+7c = 0 \quad ...(iii)$$

Solving equation (ii) and (iii), we get

$$\frac{a}{(14+6)}=\frac{-b}{(7-15)}=\frac{c}{(-2-10)}=\text{k (say)}$$

$$\frac{a}{20}=\frac{-b}{-8}=\frac{c}{-12}=\text{k}$$

$$a=20k, \quad b=8k \quad \text{and} \quad c=-12k$$

Putting the values of a, b and c in (i), we get

$$20k(x-2)+8k(y-2)\,a-12k\,(z+1) = 0$$

$$\Rightarrow \quad 4k\,[5x-10+2y-4-3z-3] = 0$$

$$\Rightarrow \quad 5x+2y-3z-17 = 0$$

This is the required equation of the plane. **Ans.**

Now, the second plane passes through the points (4, 3, 1).

Since, this plane is parallel to the above plane,

∴ D. R.'s of the second plane be < 5, 2, – 3 >

So, equation of second plane,

$$5(x-4)+2\,(y-3)-3\,(z-1) = 0$$

$$\Rightarrow \quad 5x-20+2y-6-3z+3 = 0$$

$$\Rightarrow \quad 5x+2y-3z-23 = 0 \text{ Ans.}$$

OR

Find the vector equation of the plane that contains the lines $\vec{r} = (\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}+\hat{k})$ and the point (– 1, 3, – 4). Also, find the length of the perpendicular drawn from the point (2, 1, 4) to the plane, thus obtained.

Solution :

Given, equation of the given line

$$\vec{r} = (\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}+\hat{k})$$

The plane passes through the point (– 1, 3, – 4).

Then the equation of the plane,

$$a(x+1)+b\,(y-3)+c\,(z+4) = 0 \quad ...(i)$$

Since (1, 1) lies on the plane,

$$\therefore \quad 2a-2b+4c = 0 \quad ...(ii)$$

Also, (1, 2, –1) lies on the plane

$$\therefore \quad 2a-b+3c=0 \quad ...(iii)$$

Solving equations (ii) and (iii), we get

$$\frac{a}{(-6+4)}=\frac{-b}{(6-8)}-\frac{c}{(-2+4)}=k\text{ (say)}$$

$$\frac{a}{-2}=\frac{-b}{-2}=\frac{c}{2}=k$$

$$a=-2k, \; b=2k \text{ and } c=2k$$

Putting the values of a, b, and c in (i), we get

$$-2k(x+1)+2k(y-3)+2k\,(z+4) = 0$$

$$\Rightarrow \quad -(x+1)+(y-3)+(z+4) = 0$$

$$\Rightarrow \quad -x-1+y-3+z+4 = 0$$

$$\Rightarrow \quad -x+y+z = 0$$

∴ Vector equation of plane is,

$$\vec{r}\,.(-\vec{i}+\vec{j}+\vec{k}) = 0 \text{ Ans.}$$

$$\text{Perpendicular distance} = \left|\frac{\vec{a}\,.\,\vec{n}-d}{|\vec{n}|}\right|$$

Perpendicular distance

$$=\left|\frac{\left(2\vec{i}+\vec{j}+4\vec{k}\right).\left(-\vec{i}+\vec{j}+\vec{k}\right)-0}{\sqrt{1+1+1}}\right|$$

$$=\left|\frac{-2+1+4}{\sqrt{3}}\right|$$

$$=\frac{3}{\sqrt{3}}=\sqrt{3}\text{ units.} \qquad \textbf{Ans.}$$

28. A manufacturer has three machine operators A, B and C. The first operator A produces 1% of defective items, whereas the other two operators B and C produces 5% and 7% defective items respectively. A is on the job for 50% of the time, B on the job 30% of the time and C on the job for 20% of the time. All the items are put into one stockpile and then one item is chosen at random from his and is found to be defective. What is the probability that it was produces by A? [6]

Solution :

Let H_1 be the event items produced by A

H_2 be the event items produced by B

H_3 be the event items produced by C

$$P(H_1) = \frac{50}{100}, P(H_2) = \frac{30}{100} \text{ and } P(H_3) = \frac{20}{100}$$

Let E be the event items found to be defective.

$$P\left(\frac{E}{H_1}\right) = \frac{1}{100}, P\left(\frac{E}{H_2}\right) = \frac{5}{100} \text{ and } P\left(\frac{E}{H_3}\right) = \frac{7}{100}$$

Using Bayes' theorem,

$$P\left(\frac{H_1}{E}\right) = \frac{P(H_1)P(E/H_1)}{P(H_1)P(E/H_1) + P(H_2)P(E/H_2) + P(H_3)P(E/H_3)}$$

$$= \frac{\frac{50}{100} \times \frac{1}{100}}{\frac{50}{100} \times \frac{1}{100} + \frac{30}{100} \times \frac{5}{100} + \frac{20}{100} \times \frac{7}{100}}$$

$$= \frac{50}{50 + 150 + 140} = \frac{50}{340} = \frac{5}{34}$$ **Ans.**

29. A manufacturer has employed 5 skilled men and 10 semi-skilled men and makes two models A and B of an article. The making two one item of model A requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. One item of model B requires 1 hour by a skilled man and 3 hours by a semi-skilled man. No man is expected to work more than 8 hours per day. The manufacturer's profit on an item of model A is ₹15 and on an item of model B is ₹ 10. How many of items of each model should be made per day in order to maximize daily profit? Formulate the above LPP and solve it graphically and find the maximum profit. [6]

Solution :

Let, x be the number of items of model A and y be the number of items of model B

Let Z be the required profit.

Subject to constraints :

$$2x + y \leq 8 \times 5$$

$$\Rightarrow \quad 2x + y \leq 40$$

$$2x + 3y \leq 8 \times 10$$

$$\Rightarrow \quad 2x + 3y \leq 80$$

$$x \geq 0, y \geq 0$$

Maximise $Z = 15x + 10y$

Changing the above inequalities into equations,

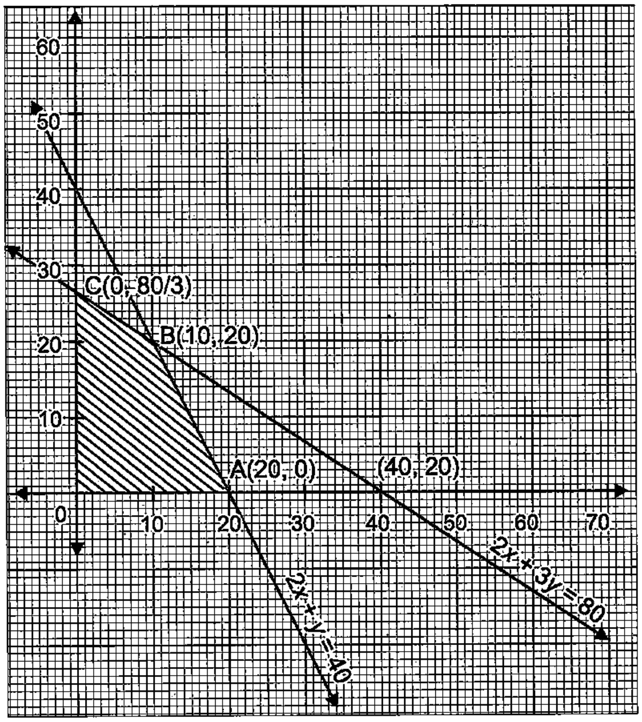
$2x + y = 40$

x	0	20
y	40	0

$2x + 3y = 80$

x	0	40
y	80/3	0

Now, ploting these points on the graph.



The shaded region is the required feasible region.

Vertices	Maximum $Z = 15x + 10y$
O (0,0)	$15 \times 0 + 10 \times 0 = 0$
A (20, 0)	$15 \times 20 + 10 \times 0 = 300$
B (10, 20)	$15 \times 10 + 10 \times 20 = 350$
C (0, 80/3)	$15 \times 0 + 10 \times 80/3 = 266.6$

Thus, the maximum profit is obtained when the manufacture produces 10 items of model A and 20 items of model B and the maximum profit ₹ 250. **Ans.**

Math 2019 (Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION–A

2. If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$, then find $\frac{d}{dx}(fog)(x)$. [1]

Solution :

Given, $f(x) = x + 7$

and $y(x) = x - 7$

Given, $(fog)(x) = f(g(x))$

$= f(x-7)$

$= (x-7)+7$

$= x$

Now, Differentiating w.r.t, x, we get

$$\frac{d(fog)}{dx}(x) = 1$$ **Ans.**

3. Find the value of $x - y$, if

$$2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}.$$ [1]

Solution :

Given, $2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$

$$\begin{bmatrix}2 & 6\\0 & 2x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

$$\begin{bmatrix}2+y & 6\\1 & 2x+2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$$

By Definition of equality, we have

$2+y = 5$ $\quad$ $2x+2 = 8$

$y = 3$ $\quad$ $2x = 6$

$x = 3$

$\therefore$ $x-y = 3-3$

$\Rightarrow$ $x-y = 0$ **Ans.**

SECTION–B

6. If $A = \begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}$, then find $(A^2 - 5A)$. [1]

Solution :

Given, $A = \begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}$

Now, $A^2 = \begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\2 & 1 & 3\\1 & -1 & 0\end{bmatrix}$

$$A^2 = \begin{bmatrix}4+0+1 & 0+0-1 & 2+0+0\\4+2+3 & 0+1-3 & 2+3+0\\2-2+0 & 0-1-0 & 1-3+0\end{bmatrix}$$

$$A^2 = \begin{bmatrix}5 & -1 & 2\\9 & -2 & 5\\0 & -1 & -2\end{bmatrix}$$

and $5A = 5\begin{bmatrix}2 & 0 & 1\\2 & -1 & 3\\1 & -1 & 0\end{bmatrix}=\begin{bmatrix}10 & 0 & 5\\10 & 5 & 15\\5 & -5 & 0\end{bmatrix}$

$\therefore$ $A^2 - 5A = \begin{bmatrix}2 & -1 & 2\\9 & -2 & 5\\0 & -1 & -2\end{bmatrix}-\begin{bmatrix}10 & 0 & 5\\10 & 5 & 15\\5 & -5 & 0\end{bmatrix}$

$$A^2 - 5A = \begin{bmatrix}-5 & -1 & -3\\-1 & -7 & -10\\-5 & 4 & -2\end{bmatrix}$$ **Ans.**

12. Find : $\int \frac{\tan^2 x \sec^2 x}{1-\tan^6 x}dx$. [1]

Solution :

Let $I = \int \frac{\tan^2 x \sec^2 x}{1-\tan^6 x}dx$

$\Rightarrow$ $I = \int \frac{\tan^2 x \sec^2 x}{1-(\tan^3 x)^2}dx$

Put $\tan^3 x = t$

$3\tan^2 x \sec^2 x\, dx = dt$

$\tan^2 x \cdot \sec^2 dx = \frac{dt}{3}$

$\Rightarrow$ $I = \frac{1}{3}\int\frac{dt}{1-t^2}$

$\Rightarrow$ $I = \frac{1}{3}\times\frac{1}{2\times 1}\log\left|\frac{1+t}{1-t}\right|+C$

$\Rightarrow$ $I = \frac{1}{6}\log\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right|+C$ **Ans.**

SECTION–C

13. Solve for x : $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$. **[2]**

Solution :

Given, $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

$$\tan^{-1}\left[\frac{2x+3x}{1-(2x)(3x)}\right] = \frac{-\pi}{4}$$

$$\left[\because \tan^{-1}x + \tan^{-1}y = \tan^{1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \tan\frac{\pi}{4}$$

$$\Rightarrow \left(\frac{5x}{1-6x^2}\right) = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{1-6x^2} = 1$$

$$\Rightarrow 5x = 1-6x^2$$

$$\Rightarrow 6x^2 + 5x - 1 = 0$$

$$\Rightarrow 6x^2 + 6x - x - 1 = 0$$

$$\Rightarrow (6x-1)(x+1) = 0$$

$$\Rightarrow x = \frac{1}{6} \text{ or } x = -1$$

$$\therefore x = \frac{1}{6}$$ **Ans.**

18. Using properties of determinants, prove the following :

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a).$$ **[4]**

Solution :

Taking L. H. S.

$$= \begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$, we get

$$= \begin{vmatrix} a+b+c & -c & -b \\ a+b & a+b & -(a+b) \\ a+c & -(a+c) & a+c \end{vmatrix}$$

Taking $(a + b)$ common from R_2 and $(a + c)$ common from R_3, we get

$$= (a+b)(a+c)\begin{vmatrix} a+b+c & -c & -b \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

Now, Applying $C_1 \to C_2 + C_2$, we get

$$= (a+b)(a+c)\begin{vmatrix} (a+b) & -c & -b \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

Applying $C_1 \to C_2 + C_3$, we get

$$= (a+b)(a+c)\begin{vmatrix} (a+b) & -(b+c) & -b \\ 2 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

Now, Expanding along R_3, we get

$$= (a+b)(a+c)[1.\{0 + 2(b+c)\}]$$

$$= 2(a+b)(a+c)[2.(b+c)]$$

$$= 2(a+b)(b+c)(c+a) = \text{R.H.S.}$$

Hence Proved.

19. If $x = \cos t + \log\tan\left(\frac{t}{2}\right)$, $y = \sin t$, then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. **[4]**

Solution :

Given,

$$x = \cos t + \log\tan\left(\frac{t}{2}\right)$$

Differentiating w.r.t. t, we get

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}.\frac{1}{2}\sec^2\frac{t}{2}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{\cos\left(\frac{t}{2}\right)}{2.\sin\left(\frac{t}{2}\right)}.\frac{1}{\cos^2\left(\frac{t}{2}\right)}$$

$$\Rightarrow \frac{dx}{dt} = -\sin t + \frac{1}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}$$

$$\frac{dx}{dt} = -\sin t + \frac{1}{\sin t}$$

$$\left[\because A = 2\sin\frac{A}{2}\cos A\frac{A}{2}\right]$$

$$\Rightarrow \frac{dx}{dt} = \frac{-\sin^2 t + 1}{\sin t}$$

$$\Rightarrow \frac{dx}{dt} = -\frac{\cos^2 t}{\sin t} \quad \text{...(i)}$$

and $y = \sin t$ (Given)

Differentiating w.r.t., we get

$$\frac{dy}{dt} = \cos t$$

Now, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \cos t \times \frac{\sin t}{\cos^2 t}$

$\Rightarrow \quad \frac{dy}{dx} = \tan t$

Again, Differentiating both sides w.r.t., x, we get

$$\frac{d^2y}{dx^2} = \sec^2 t \frac{dt}{dx}$$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 t \times \frac{\sin t}{\cos^2 t}$ [using (i)]

$$\Rightarrow \quad \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}} = \sec^2 \pi/4 . \frac{\sin \pi/4}{\cos^2 \pi/4}$$

$$= \left(\sqrt{2}\right)^2 . \frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{2 \times \frac{1}{\sqrt{2}}}{\frac{1}{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$ **Ans.**

SECTION–D

24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of a radius r is $\frac{4r}{3}$. Also find the maximum volume of cone. [6]

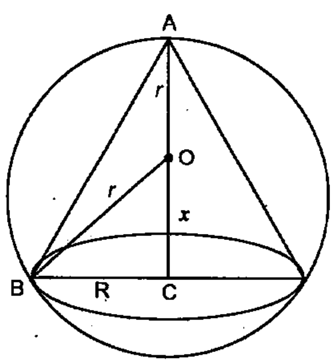
Solution :

Let R be the radius of cone. Let OA = OB = r (radius of sphere)

$$AC = r + x \quad ...(i)$$

(height of cone)

Let V be the volume of cone.



To prove : $AC = \frac{4r}{3}$

Now, $V = \frac{\pi}{3}(BC)^2(AC)$ ...(ii)

(from the figure)

$$V = \frac{\pi}{3}(BC)^2(r+x)$$

Now in ΔOBC,

$$(OB)^2 = (OC)^2 + (BC)^2$$

$$r^2 = x^2 + R^2$$

$$r^2 - x^2 = (BC)^2 \quad ...(iii)$$

$\therefore \quad V = \frac{\pi}{3}(r^2 - x^2)(r + x)$

$$V = \frac{\pi}{3}\underset{\text{I}}{(r + x)^2}\ \underset{\text{II}}{(r - x)}$$

Differentiating both sides w. r. t. x, we get

$$\frac{dV}{dx} = \frac{\pi}{3}[(r + x)^2(-1) + (r - x).2(r + x)]$$

$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}(r + x)[-r - x + 2r - 2x]$

$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}(r + x)(r - 3x)$

For maximum or minimum volume,

$$\frac{dV}{dx} = 0$$

$\Rightarrow \quad \frac{\pi}{3}(r + x)(r - 3x) = 0$

$\Rightarrow \quad x = r$ or $x = \frac{r}{3}$

$x = r$ is not possible.

Now Again differentiating w. r. t. x_1, we get

$$\frac{d^2V}{dx^2} = \frac{\pi}{3}[(r+x)(-3)+(r-3x)(1)]$$

$\Rightarrow \quad \frac{d^2V}{dx^2} = \frac{\pi}{3}[-3r - 3x + r - 3x]$

$\Rightarrow \quad \frac{d^2V}{dx^2} = \frac{\pi}{3}(-2r - 6x)$

$\Rightarrow \quad \left.\frac{d^2V}{dx^2}\right|_{x=\frac{r}{3}} = \frac{\pi}{3}\left(-2r - \frac{6r}{3}\right)$

$$= \frac{-4\pi r}{3} < 0$$

So, Volume of cone is maximum when $x = \frac{r}{3}$.

Now, Put $x = \frac{r}{3}$ in (i)

$$AC = r + x$$
$$= r + \frac{r}{3}$$
$$AC = \frac{4r}{3}$$

The altitude of cone is $\frac{4r}{3}$ when its volume is maximum. **Hence Proved.**

From equation (iii),

$$(BC)^2 = r^2 - \left(\frac{r}{3}\right)^2 = r^2 - \frac{r^2}{9}$$

$$(BC)^2 = \frac{8r^2}{9}$$

Maximum volume of cone $= \frac{\pi}{3}\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right)$

[Using (ii)]

$$= \frac{32\pi r^3}{81}$$ **Ans.**

25. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1}. Hence solve the following system of equations: $2x-3y+5z=11$, $3x+2y-4z=-5, x+y-2z=-3$. [6]

Solution :

Given, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$$|A| = 2(-4+4) + 3(-6+4) + 5(3-2)$$
$$|A| = 3(-2) + 5(1)$$
$$|A| = -1 \neq 0$$

$\therefore$ A^{-1} exists.

Cofactors of A,

$a_{11} = 0,\quad a_{21} = -1,\quad a_{31} = 2,$
$a_{12} = 2,\quad a_{22} = -9,\quad a_{32} = 23,$
$a_{13} = 1,\quad a_{23} = -5,\quad a_{33} = 13,$

$$\text{adj } A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$ ...(i)**Ans.**

Now, given equations are

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Here, $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$$AX = B$$

$\therefore$ $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$ [From (i)]

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1,\ y = 2$ and $z = 3$ **Ans.**

OR

Obtain the inverse of the following matrix using elementary operations :

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution :

Given, $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$$A = IA$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$

Applying $R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix}1 & 2 & 3\\-1 & 1 & 2\\3 & 1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}A$$

Applying $R_2 \rightarrow R_2 + R_1$, we get

$$\begin{bmatrix}1 & 2 & 3\\0 & 3 & 5\\3 & 1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}A$$

Applying $R_3 \rightarrow R_3 - 3R_1$, we get

$$\begin{bmatrix}1 & 2 & 3\\0 & 3 & 5\\0 & -5 & -8\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & 1 & 0\\0 & -3 & 1\end{bmatrix}A$$

Applying $R_2 \rightarrow \frac{1}{3}R_2$, we get

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & \frac{5}{3}\\0 & -5 & -8\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\\frac{1}{3} & \frac{1}{3} & 0\\0 & -3 & 1\end{bmatrix}A$$

Applying $R_1 \rightarrow R_1 - 2R_2$, we get

$$\begin{bmatrix}1 & 0 & \frac{-1}{3}\\0 & 1 & \frac{5}{3}\\0 & -5 & -8\end{bmatrix} = \begin{bmatrix}\frac{-2}{3} & \frac{1}{3} & 0\\\frac{1}{3} & \frac{1}{3} & 0\\0 & -3 & 1\end{bmatrix}A$$

Applying $R_3 \rightarrow R_3 + 5R_2$, we get

$$\begin{bmatrix}1 & 0 & \frac{-1}{3}\\0 & 1 & \frac{5}{3}\\0 & 0 & \frac{1}{3}\end{bmatrix} = \begin{bmatrix}\frac{-2}{3} & \frac{1}{3} & 0\\\frac{1}{3} & \frac{1}{3} & 0\\\frac{5}{3} & \frac{-4}{3} & 1\end{bmatrix}A$$

Applying $R_1 \rightarrow R_1 - R_3$, we get

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & \frac{5}{3}\\0 & 0 & \frac{1}{3}\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\\frac{1}{3} & \frac{1}{3} & 0\\\frac{5}{3} & \frac{-4}{3} & 1\end{bmatrix}A$$

Applying $R_2 \rightarrow R_2 - 5R_3$, we get

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & \frac{1}{3}\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\-8 & 7 & -5\\\frac{5}{3} & \frac{-4}{3} & 1\end{bmatrix}A$$

Now, Applying $R_3 \rightarrow 3R_3$, we get

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\-8 & 7 & -5\\5 & -4 & 3\end{bmatrix}A$$

Hence, $$A^{-1} = \begin{bmatrix}1 & -1 & 1\\-8 & 7 & -5\\5 & -4 & 3\end{bmatrix}$$ **Ans.**

Math 2019 (Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION-A

1. If $3A - B = \begin{bmatrix}5 & 0\\1 & 1\end{bmatrix}$ and $B = \begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}$, then find the matrix A. **[1]**

Solution :

Given, $$B = \begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}$$

Let $$C = \begin{bmatrix}5 & 0\\1 & 1\end{bmatrix}$$

$\therefore$ $3A - B = C$

$\Rightarrow$ $A = \frac{1}{3}(B + C)$

$\Rightarrow$ $A = \frac{1}{3}\left(\begin{bmatrix}4 & 3\\2 & 5\end{bmatrix} + \begin{bmatrix}5 & 0\\1 & 1\end{bmatrix}\right)$

$\Rightarrow$ $A = \frac{1}{3}\begin{bmatrix}9 & 3\\3 & 6\end{bmatrix}$

$\therefore$ $A = \begin{bmatrix}3 & 1\\1 & 2\end{bmatrix}$ **Ans.**

2. Write the order and the degree of the following differential equation :

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$ **[1]**

Solution :

Given,

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0$$

$\therefore$ Order of the given differential euqation is 2 and degree is 2. **Ans.**

SECTION-B

5. Find : $\int \sin x . \log \cos x \, dx$. **[2]**

Solution :

Let $\quad I = \int \sin x \log \cos x \, dx$

Put $\cos x = t$

$$\sin x \, dx = -dt$$

$$I = \int \underset{I}{\log t} . \underset{II}{1} \, dt$$

$$\Rightarrow \quad I = \left\{\log . t - \int \frac{1}{t} . t dt\right\}$$

[Using By Parts]

$$\Rightarrow \quad I = -t \log t + t + c$$

$$\Rightarrow \quad I = -\cos x (\log \cos x - 1) + C$$

Ans.

6. Evaluate : $\int_{-\pi}^{\pi} (1-x^2) \sin x \cos^2 x \, dx$. **[2]**

Solution :

Let $\quad I = \int_{-\pi}^{\pi} (1-x^2) \sin x \cos^2 x \, dx$

Let $\quad f(x) = (1-x^2) \sin x \cos^2 x$

Then $\quad f(-x) = -(1-x^2) \sin x \cos^2 x$

$$\begin{bmatrix} \because \cos(-\theta) = \cos\theta \\ \sin(-\theta) = -\sin\theta \end{bmatrix}$$

$\therefore \quad f(x) = -f(x)$

So, $f(x)$ is odd.

$\therefore \int_{-a}^{a} f(x) dx = 0$

when $f(x)$ is odd

$\therefore \quad I = 0$ **Ans.**

OR

Evaluate : $\int_{-1}^{2} \frac{|x|}{x} dx$.

Solution :

Let $\quad I = \int_{-1}^{2} \frac{|x|}{x} dx$

$$I = \int_{-1}^{0} \frac{-x}{x} dx + \int_{0}^{2} \frac{x}{x} dx$$

$$\Rightarrow \quad I = \int_{-1}^{0} -1 \, dx + \int_{0}^{2} 1 \, dx$$

$$\Rightarrow \quad I = -[x]_{-1}^{0} + 1[x]_0^2$$

$$\Rightarrow \quad I = -[0+1] + [2-0]$$

$$\Rightarrow \quad I = -1+2$$

$$\Rightarrow \quad I = 1$$

Ans.

8. Find a matrix A such that 2A – 3B + 5C = 0, where $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$. **[2]**

Solution :

Given, $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$

and $2A - 3B + 5C = 0$

$$\Rightarrow \quad 2A = -5C + 3B$$

$$\Rightarrow \quad A = \frac{1}{2}[3B - 5C]$$

$$\Rightarrow \quad A = \frac{1}{2}\left(3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}\right)$$

$$\Rightarrow \quad A = \frac{1}{2}\left(\begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} - \begin{bmatrix} 10 & 0 & -10 \\ 35 & 5 & 30 \end{bmatrix}\right)$$

$$\Rightarrow \quad A = \frac{1}{2}\begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$$

Ans.

SECTION–C

13. Using propeties of determinants, prove the following :

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

[4]

Solution :

$$\text{L.H.S.} = \begin{vmatrix} a & b & c \\ a+b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$, we get

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Taking $(a+b+c)$ from R_1, we get

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ a-b & 2b-c-a & c-2a+b \\ b+c & a-b & a-c \end{vmatrix}$$

Now, Expanding along R_1, we get

$$= (a+b+c)\,[(1)\,\{(2b-c-a)\,(a-c)-(a-b)\,(c-2a+b)\}]$$

$$= (a+b+c)\,[(2ba-2bc-ac+c^2-a^2+ac) - (ac-2a^2+ab-bc+2ab-b^2)]$$

$$= (a+b+c)\,[2ba-2bc+c^2-a^2-ac+2a^2 - ab+bc-2ab+b^2]$$

$$= (a+b+c)\,[a^2+b^2+c^2-ab-bc-ca]$$

$$= a^3+b^3+c^3-3abc = \text{R.H.S.}$$

Hence Proved.

20. Find : $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$ **[4]**

Solution :

Let $\quad I = \int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$

Put $\quad \sin x = t$

$\cos x\, dx = dt$

$\therefore \quad I = \int \frac{dt}{(1+t)(2+t)}$

Let $\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}$

$$1 = A(2+t) + B(1+t)$$

Comparing coefficient of t

$$0 = A + B$$

$$A = -B \quad \text{...(i)}$$

Comparing constant terms

$$1 = 2A + B$$

$$1 = -2B + B$$

$$-1 = B$$

$$\therefore \quad A = 1$$

$$\frac{1}{(1+t)(2+t)} = \frac{1}{(1+t)} - \frac{1}{(2+t)}$$

Now, $\quad I = \int \frac{dt}{1+t} - \int \frac{dt}{2+t}$

$\Rightarrow \quad I = \log|1+t| - \log|2+t| + c$

$\Rightarrow \quad I = \log\left|\frac{1+t}{2+t}\right| + C$

$\left[\because \log m - \log n = \log\left(\frac{m}{n}\right)\right]$

$\Rightarrow \quad I = \log\left|\frac{1+\sin x}{2+t}\right| + C$ **Ans.**

21. Solve the differential equation :

$\frac{dy}{dx} - \frac{2x}{1+x^2}y = x^2 + 2.$ **[4]**

Solution :

Given, $\frac{dy}{dx} - \frac{2x}{1+x^2}y = x^2+2$

Here, $\quad P = -\frac{2x}{1+x^2}$ and $Q = x^2 + 2$

$$\text{I.F.} = e^{\int P dx} = e^{\int -\frac{2x}{1+x^2}dx}$$

$$\text{I.F.} = \frac{1}{1+x^2}$$

$\therefore$ The solution of given differential equation is

$$y(\text{I.F}) = \int(\text{I.F.} \times Q)\,dx$$

$\Rightarrow \quad y\left(\frac{1}{1+x^2}\right) = \int\frac{x^2+2}{1+x^2}dx$

$\Rightarrow \quad y\frac{1}{1+x^2} = \int\left(1+\frac{1}{1+x^2}\right)dx$

$\Rightarrow \quad y\frac{1}{1+x^2} = x + \tan^{-1}x + C$

or $\quad y = (1+x^2)\,(x + \tan + x + C)$ **Ans.**

OR

Solve the differential equations :

$(x+1)\frac{dy}{dx} = 2e^{-1} - 1;\ y(0) = 0.$ **[4]**

Solution :

Given,

$$(x+1)\,\frac{dy}{dx} = 2e^{-y} - 1$$

$$\Rightarrow \quad \frac{dy}{2e^{-y}-1} = \frac{dx}{x+1}$$

$$\Rightarrow \quad \int \frac{e^y dy}{2-e^y} = \int \frac{dx}{x+1}$$

Put $2-e^y = t$

$$-e^y dy = dt$$

$$e^y dy = -dt$$

$$\therefore \quad \int -\frac{dt}{t} = \int \frac{dx}{x+1}$$

$$\Rightarrow \quad -\log t = \log|x+1| + \log C$$

$$\Rightarrow \quad -\log t^{-1} = \log[C(x+1)]$$

$$\Rightarrow \quad \frac{1}{t} = C(x+1)$$

$$\Rightarrow \quad \frac{1}{2-e^y} = C(x+1) \qquad ...(i)$$

Put $x = 0$ and $y = 0$ in (i), we get

$$\Rightarrow \quad \frac{1}{2} = C$$

$$\therefore \quad \frac{1}{2-e^y} = \frac{1}{2}(x+1)$$ **Ans.**

SECTION–D

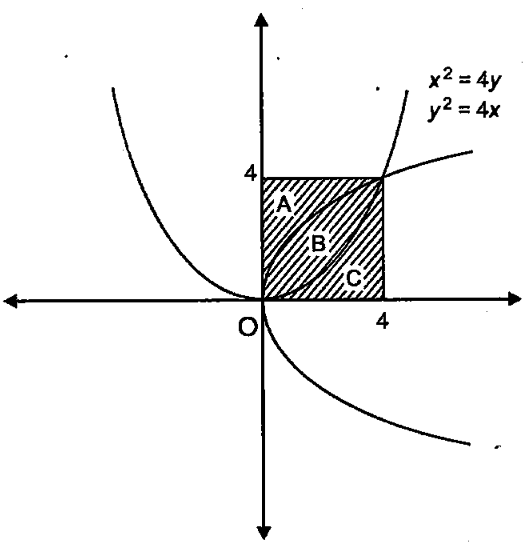
26. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the by of the squre bounded by $x = 0, x = 4, y = 4$ and $y = 0$ into three equal parts. [6]

Solution :

Given, $y^2 = 4x$ ...(i)

$x^2 = 4y$ ...(ii)

$x = 0, x = y, y = 0$ and $y = 4$



For point of interaction put $y = \frac{x^2}{4}$ in (iii)

$$\Rightarrow \quad \frac{x^4}{16} = 4x$$

$$\Rightarrow \quad x^4 = 64x$$

$$\Rightarrow \quad x^4 - 64x = 0$$

$$\Rightarrow \quad x(x^3 - 64) = 0$$

$$\Rightarrow \quad x = 0, x = 4$$

Area of part A,

$$I = \int_0^4 x \text{ of the curve I } dy$$

$$I = \int_0^4 \frac{y^2}{4} dy$$

$$I = \frac{1}{4 \times 3}\left[y^3\right]_0^4$$

$$I = \frac{1}{12}[4^3]$$

$$I = \frac{64}{12}$$

$$I = \frac{16}{3} \text{ sq. units}$$

Area of Part B,

$$I = \int_0^4 y \text{ of the I}^{\text{st}} \text{ curve } dx - \int_0^4 y \text{ of the 2}^{\text{nd}} \text{ curve } dx$$

$$I = \int_0^4 2\sqrt{x}dx - \int_0^4 \frac{x^2}{4} dx$$

$$I = 2 \times \frac{2}{3}\left[x^{3/2}\right]_0^4 - \frac{1}{12}\left[x^3\right]_0^4$$

$$I = \frac{4}{3}[4^{3/2}] - \frac{1}{12}[4^3]$$

$$I = \frac{4}{3} \times 8 - \frac{1}{12} \times 64$$

$$I = \frac{32}{3} - \frac{16}{3}$$

$$I = \frac{16}{3} \text{ sq units}$$

Area of part C,

$$I = \int_0^4 y \text{ of the 2}^{\text{nd}} \text{ curve } dx$$

$$I = \int_0^4 \frac{x^2}{4} dx$$

$$I = \frac{1}{4}\int_0^4 x^2 dx$$

$$I = \frac{1}{12}\left[x^3\right]_0^4$$

$$I = \frac{1}{12} \times 64$$

$$I = \frac{16}{3} \text{ sq. units}$$

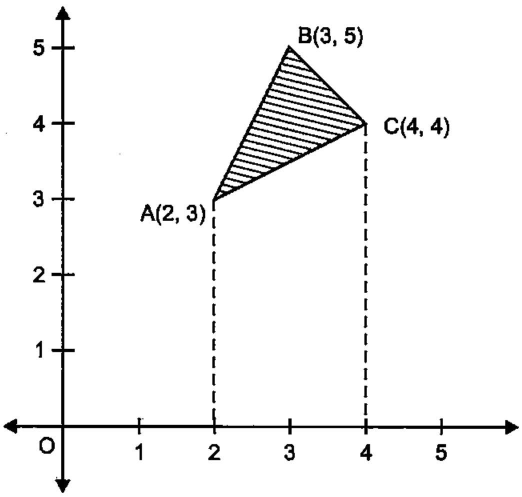
$\therefore$ The curves $y^2 = 4x$ and $x^2 = 4y$ divides the area of the square into three equal parts. **Hence Proved.**

OR

Using integration, find the area of the triangle whose vertices are (2, 3), (3, 5) and (4, 4).

Solution :

Given, the vertices of ΔABC, A(2, 3), B (3, 5) and C (4, 4).



Equation of line AB,

$$(y-3) = \frac{5-3}{3-2}(x-2)$$

$$(y-3) = 2(x-2)$$

$$y = 2x-4+3$$

$$y = 2x-1$$

Equation of line BC,

$$(y-5) = \frac{4-5}{4-3}(x-3)$$

$$(y-5) = -(x-3)$$

$$y-5 = -x-3$$

$$y = -x+8$$

Equation of line AC,

$$(y-3) = \frac{4-3}{4-2}(x-2)$$

$$(y-3) = \frac{1}{2}(x-2)$$

$$y = \frac{x-2}{2}+3$$

$$y = \frac{x+4}{2}$$

$$I = \int_2^3 (2n-1)\,dx + \int_3^4 (-x+8)\,dx - \int_2^4 \frac{x+4}{2}\,dx$$

Area of Δ ABC

$$I = \int_2^3 y_{AB}dx + \int_3^4 y_{BC}\,dx - \int_2^3 y_{AC}dx$$

$$I = [x^2]_2^3 - [x]_2^3 - \frac{1}{2}[x^2]_3^4 + 8[x]_3^4 - \frac{1}{2}\left[\frac{x^2}{2}\right]_2^4 + 2[x]_2^4$$

$$I = [9-4]-[3-2]-\frac{1}{2}[16-9] + 8[4-3]-\frac{1}{4}[16-4]+2[4-2]$$

$$I = 5-1-\frac{7}{2}+8-\frac{12}{4}+4$$

$$I = 4-\frac{7}{2}+8-3+4$$

$$I = 13-\frac{7}{2}$$

$$I = \frac{19}{2} \text{ Sq. units}$$ **Ans.**

29. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean and variance of the number of kings. [4]

Solution :

Let S = Number of kings.

and F = number of non kings

$\therefore$ $P(S) = \frac{4}{52}$ and $P(F) = \frac{48}{52}$

Let x_i be the event that king is drawn.

x_i	Px_i	x_iPx_i	x_i^2	$Px_i\,x_i^2$
0	$\frac{48}{52}\times\frac{47}{51}\times 1 = \frac{2256}{2652}$	0	0	0
1	$\frac{4}{52}\times\frac{48}{51}\times 2 = \frac{384}{2652}$	$\frac{384}{2652}$	1	$\frac{384}{2652}$
2	$\frac{4}{52}\times\frac{3}{51}\times 1 = \frac{12}{2652}$	$\frac{12}{2652}$	4	$\frac{48}{2652}$

Px_i	$2256/2652$	$384/2652$	$12/2652$
x_i	0	1	2

$$\text{Mean } (\bar{X}) = \Sigma x_i^2 P(x_i)$$

$$= \frac{384+12}{2652} = \frac{396}{2652}$$

$$\text{Variance } (\sigma^2) = \Sigma x_i^2 P(x_i) - (\bar{X})^2$$

$$= \frac{432}{2652} = \left(\frac{396}{2652}\right)^2$$

$$= \frac{36}{221} - \left(\frac{33}{221}\right)^2$$

$$= \frac{(36\times 221)-(33\times 33)}{(221\times 221)}$$

$$= \frac{7956-1089}{48841} = \frac{6867}{48841}$$ **Ans.**

