CLASS XII (2019-20)
MATHEMATICS (041)
SAMPLE PAPER-2

Time : 3 Hours **Maximum Marks : 80**

General Instructions :

(i) All questions are compulsory.
(ii) The questions paper consists of 36 questions divided into 4 sections A, B, C and D.
(iii) Section A comprises of 20 questions of 1 mark each. Section B comprises of 6 questions of 2 marks each. Section C comprises of 6 questions of 4 marks each. Section D comprises of 4 questions of 6 marks each.
(iv) There is no overall choice. However, an internal choice has been provided in three questions of 1 mark each, two questions of 6 marks each. You have to attempt only one of the alternatives in all such questions.
(v) Use of calculators is not permitted.

Section-A

DIRECTION : (Q 1-Q 10) are multiple choice type questions. Select the correct option.

1. If $f(x_1)=f(x_2)\Rightarrow x_1=x_2\,\forall\, x_1,x_2\varepsilon A$, then what type of a function if $f:A\longrightarrow B$? [1]

(a) One-one (b) Constant
(c) Onto (d) Many one

Ans : (a) One-one

If, $f(x_1)=f(x_2)$

$\Rightarrow$ $x_1=x_2$, then $f(x)$ is one-one.

2. $\sin^{-1}\dfrac{1}{x}=?$ [1]

(a) $\sec^{-1}x$ (b) $\operatorname{cosec}^{-1}x$
(c) $\tan^{-1}x$ (d) $\sin x$

Ans : (b) $\operatorname{cosec}^{-1}x$

We know that,

$$\sin^{-1}\frac{1}{x}=\operatorname{cosec}^{-1}x$$

3. $A=\begin{bmatrix}3&6\\5&-4\end{bmatrix}$, $B=\begin{bmatrix}7&8\\5&6\end{bmatrix}$, $2A+3B=?$ [1]

(a) $\begin{bmatrix}27&24\\22&10\end{bmatrix}$ (b) $\begin{bmatrix}27&36\\25&10\end{bmatrix}$
(c) $\begin{bmatrix}27&36\\25&15\end{bmatrix}$ (d) $\begin{bmatrix}27&36\\35&10\end{bmatrix}$

Ans : (b) $\begin{bmatrix}27&36\\25&10\end{bmatrix}$

We have,

$$2A+3B=2\begin{bmatrix}3&6\\5&-4\end{bmatrix}+3\begin{bmatrix}7&8\\5&6\end{bmatrix}$$
$$=\begin{bmatrix}6&12\\10&-8\end{bmatrix}+\begin{bmatrix}21&24\\15&18\end{bmatrix}$$
$$=\begin{bmatrix}27&36\\25&10\end{bmatrix}$$

4. If $\lambda\in R$ and $\Delta=\begin{vmatrix}a&b\\c&d\end{vmatrix}$ then $\lambda\Delta=$ [1]

(a) $\begin{vmatrix}\lambda a&\lambda b\\\lambda c&\lambda d\end{vmatrix}$ (b) $\begin{vmatrix}\lambda a&b\\c&d\end{vmatrix}$
(c) $\begin{vmatrix}\lambda a&b\\\lambda c&d\end{vmatrix}$ (d) None of these

Ans : (c) $\begin{vmatrix}\lambda a&b\\\lambda c&d\end{vmatrix}$

Multiplying a determinant by λ means multiplying the elements of only one row (or one column) by λ.

Hence, $\lambda\Delta=\lambda\begin{vmatrix}a&b\\c&d\end{vmatrix}=\begin{vmatrix}\lambda a&b\\\lambda c&d\end{vmatrix}$

5. $\dfrac{d}{dx}[\sin^{-1}x]=?$ [1]

(a) $\dfrac{1}{\sqrt{1-x^2}}$ (b) $\dfrac{-1}{\sqrt{1-x^2}}$
(c) $\dfrac{1}{\sqrt{1+x^2}}$ (d) $\sqrt{1-x^2}$

Ans : (a) $\dfrac{1}{\sqrt{1-x^2}}$

Let, $f(x)=\sin^{-1}x$

$$=\frac{d}{dx}f(x)$$

$$= \frac{d}{dx}(\sin^{-1}x)$$

$$= \frac{1}{\sqrt{1-x^2}}$$

6. The slope of the tangent to the curve, $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$ is- [1]

(a) $\frac{12}{7}$ (b) $\frac{-6}{7}$

(c) $\frac{6}{7}$ (d) $\frac{-12}{7}$

Ans : (c) $\frac{6}{7}$

We have,

$$t^2 + 3t - 8 = 2 \text{ and } 2t^2 - 2t - 5 = -1$$
$$t^2 + 3t - 10 = 0 \text{ and } 2t^2 - 2t - 4 = 0$$
$$(t+5)(t-2) = 0 \text{ and } t^2 - t - 2 = 0$$
$$t = -5, 2 \text{ and}$$
$$(t-2)(t+1) = 0$$
$$t = -5, 2 \text{ and } t = 2, -1$$
$$t = 2$$

Now, $\frac{dy}{dt} = 4t^2 - 2$ and $\frac{dx}{dt} = 2t + 3$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{4t^2-2}{2t+3}$$

Slope of the tangent

$$\left(\frac{dy}{dx}\right)_{(2,-1)} = \frac{4t-2}{2t+3}$$

$$= \frac{4(2)-2}{2\times 2+3} \quad [t = 2]$$

$$= \frac{8-2}{4+3} = \frac{6}{7}$$

7. $\int_0^1 \frac{dx}{1+x^2} = ?$ [1]

(a) $\frac{-\pi}{4}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) $\frac{-\pi}{2}$

Ans : (b) $\frac{\pi}{4}$

Let, $I = \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^1$

$$= \tan^{-1}1 - \tan^{-1}0$$

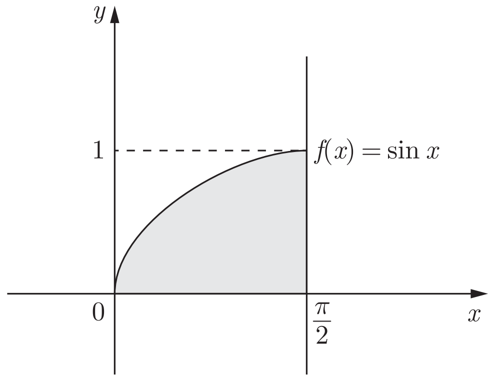
$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

8. Area between the x-axis and the curve $y = \sin x$, from $x = 0$ to $x = \frac{\pi}{2}$ is [1]

(a) 2 (b) -1

(c) 1 (d) None of these

Ans : (c) 1



9. Integrating factor of the differential equation $\frac{dy}{dx} + y\sec x = \tan x$ is- [1]

(a) $\sec x + \tan x$ (b) $\sec x - \tan x$

(c) $\sec x$ (d) $\tan x \sec x$

Ans : (a) $\sec x + \tan x$

We have,

$$\frac{dy}{dx} + y\sec x = \tan x$$

$$\text{I.f.} = e^{\int \sec x\, dx} = e^{\log(\sec x + \tan x)}$$

$$= \sec x + \tan x$$

10. The position vector of the point (x, y, z) is- [1]

(a) $x\hat{i} - y\hat{j} - z\hat{k}$ (b) $x\hat{i} + y\hat{j} - z\hat{k}$

(c) $x\hat{i} - y\hat{j} + z\hat{k}$ (d) $x\hat{i} + y\hat{j} + z\hat{k}$

Ans : (d) $x\hat{i} + y\hat{j} + z\hat{k}$

Let $P(x, y, z)$. then position vector of P is given by

$$\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$$

Q. 11-15 (Fill in the blanks)

11. Let $A = \{1,2,3,4\}$ and R be the equivalence relation on $A \times A$ defined by $(a,b)R(c,d)$ if $a + d = b + c$ the equivalence class $[(1,3)]$ is .... [1]

Ans :

Here, $[(1,3)] = \{(x,y) \in A \times A : (x,y)R(1,3)\}$

$$= \{(x,y) \in A \times A : x + 3 = y + 1\}$$
$$= \{(x,y) \in A \times A : y = x + 2\}$$
$$= \{(x,y) \in A \times A : y - x = 2\}$$
$$= \{(1,3),(2,4)\}$$ **(1)**

12. If $A = [a_{ij}]$ is a matrix of order 2×2, such that $|A| = -15$ and c_{ij} represents the cofactor of a_{ij}, then $a_{21}c_{21} + a_{22} + c_{22}$ is .... [1]

Ans :

We have, $A = [a_{ij}]$ and $|A| = -15$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Now, $c_{21} = (-1)^{2+1} a_{12} = (-1)^3 a_{12}$

$= -a_{12}$ **(1/2)**

and $c_{22} = (-1)^{2+2} a_{11} = (-1)^4 a_{11}$

$= a_{11}$

$\therefore \quad a_{21}c_{21} + a_{22}c_{22} = a_{21}(-a_{12}) + a_{22}a_{11}$

$= -a_{21}a_{12} + a_{22}a_{11} = |A|$

$= -15$ **(1/2)**

13. A function $f: A \to B$ is said to be .... if every element of B is the image of some element of A under f. [1]

Ans : Surjective

or

Relation R in a set A is called an .... relation. If each element of A is related to every element of A, i.e. $R = A \times A$.

Ans : Universal

14. A corner point of a feasible region is a point in the region which is the .... of two boundary lines. [1]

Ans : Intersection

15. If $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, then $f(x)$ is an .... function. [1]

Ans : Even function

or

$\int e^x(f(x) + f'(x))\,dx$ is equal to ....

Ans :

$$\int e^x(f(x) + f'(x))\,dx = e^x f(x) + c$$

DIRECTION : (Q.no. 16-20) Answer the following questions.

16. Find the area of the parallelogram determined by the vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.[1]

Ans :

Let, $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

Then, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$

$= \hat{i}(2+6) - \hat{j}(1-9) + \hat{k}(-2-6)$

$= 8\hat{i} + 8\hat{j} - 8\hat{k}$

$\therefore$ Area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.

$A = |\vec{a} \times \vec{b}| = \sqrt{8^2 + 8^2 + (-8)^2}$

$= 8\sqrt{3}$ sq units

or

Find the value of λ, if the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar.

Ans :

Given, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, then

$[\vec{a}\,\vec{b}\,\vec{c}] = 0$

$\Rightarrow \quad \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$

$2(10 + 3\lambda) + 1(5 + 9) + 1(\lambda - 6) = 0$

$20 + 6\lambda + 14 + \lambda - 6 = 0$

$7\lambda = -28$

$\lambda = -4$

17. If $Y = \tan^{-1}\left(\frac{a+x}{1-ax}\right)$, find $\frac{dy}{dx}$. [1]

Ans :

We have, $y = \tan^{-1}\left(\frac{a+x}{1-ax}\right)$

$= \tan^{-1} a + \tan^{-1} x$

$\frac{dy}{dx} = 0 + \frac{1}{1+x^2} = \frac{1}{1+x^2}$

18. If A is symmetric, then show that $B'AB$ is symmetric matrix.

Ans :

Given, A is symmetric

$A = A'$

Now, $(B'AB)' = (AB)'(B')' = B'A'B'$

$\Rightarrow \quad (B'AB)' = B'AB$

Hence, $B'AB$ is symmetric. Hence proved

19. If $\int_0^1 (3x^2 + 2x + K)\,dx = 0$, find K. [1]

Ans :

We have, $\int_0^1 (3x^2 + 2x + K)\,dx = 0$

$[x^3 + x^2 + Kx]_0^1 = 0$

$(1 + 1 + K) - 0 = 0$

$\Rightarrow \quad K = -2$

20. If $P(A/B) > P(A)$, the prove that $P(B/A) > P(B)$. [1]

Ans :

Given, $P(A/B) > P(A)$

$$\frac{P(A \cap B)}{P(B)} > P(A)$$

$$\frac{P(A \cap B)}{P(A)} > P(B) \Rightarrow P(B/A) > P(B)$$

Section B

21. If $y = \sin^{-1}(\sqrt{x}\sqrt{1-x^2} - x\sqrt{1-x})$, then find $\frac{dy}{dx}$. [2]

Ans :

Given,

$$y = \sin^{-1}\left[\sqrt{x}\sqrt{1-x^2} - x\sqrt{1-(\sqrt{x})^2}\right]$$

Now, put $x = \sin\theta$ and $\sqrt{x} = \sin\phi$

Then,

$$y = \sin^{-1}\left[\sin\phi\sqrt{1-\sin^2\theta} - \sin\theta\sqrt{1-\sin^2\phi}\right]$$

$$= \sin^{-1}[\sin\phi\cos\theta - \cos\phi\sin\theta]$$

$$[\because 1 - \sin^2 A = \cos^2 A]$$

$$= \sin^{-1}[\sin(\phi - \theta)]$$

$$[\because \sin A\cos B - \cos A\sin B = \sin(A-B)]$$

$$= \phi - \theta = \sin^{-1}\sqrt{x} - \sin^{-1}x \quad \textbf{(1)}$$

$$\begin{bmatrix}\because x = \sin\theta \Rightarrow \theta = \sin^{-1}x \\ \text{and } \sqrt{x} = \sin\phi \Rightarrow \phi = \sin^{-1}\sqrt{x}\end{bmatrix}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{2\sqrt{x}\sqrt{1-x}} - \frac{1}{\sqrt{1-x^2}} \quad \textbf{(2)}$$

or

Find a point on the parabola $y = (x-3)^2$, where the tangent is parallel to the chord joining (3,0) and (4,1).

Ans :

Let us apply Lagrange's mean value theorem for the function $f(x) = (x-3)^2$ in the interval [3,4].

Now, $f(x)$ being a polynomial function, it is continuous on [3,4].

Also, $f'(x) = 2(x-3)$, which exists for all $x\,\varepsilon\,]3,4[$

So, $f(x)$ is differentiable on]3,4[.

Thus, both the conditions of lagrange's mean value theorem are satisfied.

So, there must exist a point $c\,\varepsilon\,]3,4[$ such that

$$f'(c) = \frac{f(4) - f(3)}{(4-3)} = 1 \quad \textbf{(1)}$$

Now, $f'(c) - 1 \Leftrightarrow 2(c-3) = 1 \Leftrightarrow c = \frac{7}{2}\varepsilon\,]3,4[$

Now, $x = \frac{7}{2}$ and $y = (x-3)^2 \Rightarrow y = \frac{1}{4}$

Thus, at the point $\left(\frac{7}{2}, \frac{1}{4}\right)$ on the given curve the tangent is parallel to the chord joining (3,0) and (4,1). **(1)**

22. If $4\sin^{-1}x + \cos^{-1}x = \pi$, then find the value of x. [2]

Ans :

We have $4\sin^{-1}x + \cos^{-1}x = \pi$

$$\Rightarrow \quad 4\sin^{-1}x + \left(\frac{\pi}{2} - \sin^{-1}x\right) = \pi$$

$$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad 4\sin^{-1}x + \frac{\pi}{2} - \sin^{-1}x = \pi \quad \textbf{(1)}$$

$$\Rightarrow \quad 3\sin^{-1}x = \pi - \frac{\pi}{2}$$

$$= \frac{2\pi - \pi}{2} = \frac{\pi}{2}$$

$$\Rightarrow \quad 3\sin^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow \quad x = \sin\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad x = \frac{1}{2}\left[\because \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

(1)

23. Find the inverse of the matrix $\begin{bmatrix}-3 & 2 \\ 5 & -3\end{bmatrix}$. Hence find the matrix P satisfying the matrix equation $P\begin{bmatrix}-3 & 2 \\ 5 & -3\end{bmatrix} = \begin{bmatrix}1 & 2 \\ 2 & -1\end{bmatrix}$. [2]

Ans :

Let $A = \begin{bmatrix}-3 & 2 \\ 5 & -3\end{bmatrix}$ ...(i)

We know that

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix}C_{11} & C_{12} \\ C_{21} & C_{22}\end{bmatrix}^T$$

Now, $|A| = \begin{vmatrix}-3 & 2 \\ 5 & -3\end{vmatrix} = (-3)(-3) - (2)(5)$

$$= 9 - 10 = -1$$

$|A| = -1 > 0$, which shows that inverse of matrix exists. **(1/2)**

$$C_{11} = (-1)^{1+1} a_{22} = (-1)^2 a_{22} = a_{22} = -3$$

$$C_{12} = (-1)^{1+2} a_{21} = (-1)^3 a_{21} = -5$$

$$C_{21} = (-1)^{2+1} a_{12} = (-1)^3 a_{12} = -2$$

$$C_{22} = (-1)^{2+2} a_{11} = (-1)^4 a_{11} = -3$$ **(1/2)**

$$\therefore \quad A^{-1} = \frac{1}{(-1)}\begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}^T = \frac{1}{(-1)}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$ **(1/2)**

Now, $P\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ [given]

$\Rightarrow \quad PA = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ [from eq. (i)]

On post-multiplying both sides by A^{-1}, we get

$$P = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$[\because AA^{-1} = I \text{ and } PI = P]$

$$= \begin{bmatrix} 3+10 & 2+6 \\ 6-5 & 4-3 \end{bmatrix} = \begin{bmatrix} 13 & 8 \\ 1 & 1 \end{bmatrix}$$ **(1/2)**

24. Prove that [2]

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \left(\frac{\pi}{4} + \frac{x}{2}\right), x\,\varepsilon\left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

Ans :

We have,

$$\text{LHS} = \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$$

$$= \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2} - x\right)}{1 - \cos\left(\frac{\pi}{2} - x\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right]$$ **(1)**

$$= \tan^{-1}\left[\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)\right] = \tan^{-1}\left[\tan\left\{\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{x}{2}\right)\right\}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$$

$$= \left(\frac{\pi}{4} + \frac{x}{2}\right) = \text{RHS}$$ **(1)**

Hence proved

or

Solve for $x : \tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x, x > 0.$

Ans :

We have,

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$$

$$2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$$

$$2[\tan^{-1}1 - \tan^{-1}x] = \tan^{-1}x$$

$$2\left(\frac{\pi}{4} - \tan^{-1}x\right) = \tan^{-1}x$$ **(1)**

$$\frac{\pi}{2} - 2\tan^{-1}x = \tan^{-1}x$$

$$\Rightarrow \quad \frac{\pi}{2} = 3\tan^{-1}x$$

$$\tan^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow \quad x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

Hence, $x = \frac{1}{\sqrt{3}}$ **(1)**

25. Prove that if $\frac{1}{2} \le x \le 1$, then

$$\cos^{-1}x + \cos^{-1}\left[\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right] = \frac{\pi}{3}.$$ [2]

Ans :

Let $\cos^{-1}x = \theta$. Then, for all $x\,\varepsilon\left[\frac{1}{2}, 1\right]$, $\theta\,\varepsilon\left[0, \frac{\pi}{3}\right]$

Thus, $x = \cos\theta$ **(1/2)**

We have, $\cos^{-1}x + \cos^{-1}\left[\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right] = \frac{\pi}{3}$

$$\text{LHS} = \theta + \cos^{-1}\left[\frac{\cos\theta}{2} + \frac{\sqrt{3-3\cos^2\theta}}{2}\right]$$ **(1/2)**

$$= \theta + \cos^{-1}\left[\frac{\cos\theta}{2} + \frac{\sqrt{3}\sqrt{1-\cos^2\theta}}{2}\right]$$

$$= \theta + \cos^{-1}\left[\frac{\cos\theta}{2} + \frac{\sqrt{3}\sqrt{\sin^2\theta}}{2}\right]$$

$[\because 1 - \cos^2\theta = \sin^2\theta]$

$$= \theta + \cos^{-1}\left[\frac{\cos\theta}{2} + \frac{\sqrt{3}\sin\theta}{2}\right]$$

$$= \theta + \cos^{-1}\left[\cos\theta\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\sin\theta\right]$$ **(1/2)**

$$= \theta + \cos^{-1}\left[\cos\theta \cdot \cos\frac{\pi}{3} + \sin\frac{\pi}{3}\sin\theta\right]$$

$$= \theta + \cos^{-1}\left[\cos\left(\theta - \frac{\pi}{3}\right)\right]$$

$[\because \cos A \cos B + \sin A \sin B = \cos(A - B)]$

$$= \theta + \cos^{-1}\left[\cos\left(\frac{\pi}{3} - \theta\right)\right]$$

$$= \theta + \frac{\pi}{3} - \theta \qquad [\because \cos(-\theta) = \cos\theta]$$

$$= \frac{\pi}{3} = \text{RHS}$$ Hence Proved **(1/2)**

26. Find the approximate change in the value of $\frac{1}{x^2}$, when x changes from $x = 2$ to $x = 2.002$. [2]

Ans :

Let $y = \frac{1}{x^2}$

On differentiating both sides w.r.t x, we get

$$\frac{dy}{dx} = \frac{-2}{x^3}$$ **(1/2)**

Now, we have to find the change from $x = 2$ to $x = 2.002$.

$$\therefore \quad dy = \left(\frac{dy}{dx}\right)_{x=2} \times \Delta x$$

$$= \left(\frac{-2}{x^3}\right)_{x=2} \times (0.002)$$

$$= \frac{-2}{2^3} \times 0.002$$ **(1/2)**

$$= \frac{-2}{8} \times \frac{2}{1000}$$

$$\Rightarrow \quad dy = -0.0005$$

Thus, y decreased by 0.0005. **(1)**

Section C

27. If $\Delta = \begin{vmatrix} 1 & a & a^2 \\ a & a^2 & 1 \\ a^2 & 1 & a \end{vmatrix} = -4$, then find the value of $\begin{vmatrix} a^3-1 & 0 & a-a^4 \\ 0 & a-a^4 & a^3-1 \\ a-a^4 & a^3-1 & 0 \end{vmatrix}$. [4]

Ans :

We have,

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ a & a^2 & 1 \\ a^2 & 1 & a \end{vmatrix} = -4$$

From this, we can find out that

$$C_{11} = (-1)^{1+1}(a^3 - 1) = (-1)^2(a^3 - 1)$$

$$= a^3 - 1$$

$$C_{12} = (-1)^{1+2}(a^2 - a^2) = (-1)^3(0) = 0$$

$$C_{13} = (-1)^{1+3}(a - a^4) = a - a^4$$

$$C_{21} = (-1)^{2+1}(a^2 - a^2) = 0$$

$$C_{22} = (-1)^{2+2}(a - a^4) = (a - a^4)$$

$$C_{23} = (-1)^{2+3}(1 - a^3) = (-1)^5(1 - a)^3$$

$$= a^3 - 1$$

$$C_{31} = (-1)^{3+1}(a - a^4) = a - a^4$$

$$C_{32} = (-1)^{3+2}(1 - a^3) = a^3 - 1$$

$$C_{33} = (-1)^{3+3}(a^2 - a^2) = 0$$ **(2)**

$$\Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a^3-1 & 0 & a-a^4 \\ 0 & a-a^4 & a^3-1 \\ a-a^4 & a^3-1 & 0 \end{vmatrix}$$

We know that,

$$\begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$$ **(1)**

$$\therefore \quad \Delta_1 = \Delta^2 = (-4)^2 = 16$$

Thus, $$\begin{vmatrix} a^3-1 & 0 & a-a^4 \\ 0 & a-a^4 & a^3-1 \\ a-a^4 & a^3-1 & 0 \end{vmatrix} = 16$$ **(1)**

28. Find a and b, if the function given by $f(x) = \begin{cases} ax^2 + b, & \text{if } x < 1 \\ 2x + 1, & \text{if } x \geq 1 \end{cases}$ is differentiable at $x = 1$. [4]

Ans :

We have, $f(x) = \begin{cases} ax^2 + b, & \text{if } x < 1 \\ 2x + 1, & \text{if } x \geq 1 \end{cases}$ is differential at $x = 1$.

$\therefore$ $f(x)$ is also continuous at $x = 1$

Hence, $$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+}(2x + 1)$$

$$= 2(1) + 1 = 3$$

and $$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(ax^2 + b)$$ **(1)**

$$= a(-1)^2 + b = a + b$$

Thus, $f(1) = 3$ **(1/2)**

As $f(x)$ is continuous at $x = 1$.

$\therefore \quad a + b = 3$ ...(i) **(1/2)**

$$Lf'(1) = \lim_{h \to 0^-} \frac{f(1 - h) - f(1)}{-h}$$

$$= \lim_{h \to 0^-} \frac{a(1-h)^2 + b - 3}{-h} \quad \textbf{(1/2)}$$

$$= \lim_{h \to 0^-} \frac{a + ah^2 - 2ah + b - 3}{-h}$$

$$= \lim_{h \to 0^-} (-ah + 2a) = 2a$$

[from Eq.(i)] **(1/2)**]

$$Rf'(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0^+} \frac{2(1+h) + 1 - 3}{h}$$

$$= \frac{2(1+h) - 2}{h}$$

$$= \frac{2(1+h-1)}{h}$$

$$= \frac{2h}{h} = 2 \quad \textbf{(1/2)}$$

As $f(x)$ is differentiable at $x = 1$,

so $Lf'(1) = Rf'(1)$.

So, we have, $2a = 2 \Rightarrow a = 1$.

$\therefore$ $a = 1$ and $b = 2$ [from Eq. (i)] **(1/2)**

or

Determine the values of a and b such that the following function is continuous at $x = 0$.

$$f(x) = \begin{cases} \dfrac{x + \sin x}{\sin(a+1)x}, & \text{if } -\pi < x < 0 \\ 2, & \text{if } x = 0 \\ \dfrac{2(e^{\sin bx} - 1)}{bx}, & \text{if } x > 0 \end{cases}$$

Ans :

$$\text{We have, } f(x) = \begin{cases} \dfrac{x + \sin x}{\sin(a+1)x}, & \text{if } -\pi < x < 0 \\ 2, & \text{if } x = 0 \\ \dfrac{2(e^{\sin bx} - 1)}{bx}, & \text{if } x > 0 \end{cases}$$

Since, $f(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x + \sin x}{\sin(a+1)x}]$$

On multiplying and dividing denominator by $(a+1)x$, we get

$$= \frac{\lim\limits_{x \to 0^-} \dfrac{x + \sin x}{\sin(a+1)x}}{(a+1)x} \times (a+1)x$$

$$= \lim_{x \to 0^-} \frac{\dfrac{x + \sin x}{x}}{\dfrac{\sin(a+1)x}{(a+1)x} \cdot (a+1)} \quad \textbf{(1)}$$

$$= \lim_{x \to 0^-} \frac{1 + \dfrac{\sin x}{x}}{\dfrac{\sin(a+1)x}{(a+1)x} \cdot (a+1)}$$

$$= \frac{1+1}{1 \cdot (a+1)}$$

$$= \frac{2}{a+1} \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{2(e^{\sin bx} - 1)}{bx}$$

On multiplying and dividing by $\sin bx$, we get

$$\lim_{x \to 0^+} \frac{2(e^{\sin bx} - 1)}{\sin bx} \times \frac{\sin bx}{bx} = 2 \times 1 \times 1$$

$$= 2$$

$$\left[\because \lim_{x \to 0} \frac{e^{\sin x} - 1}{\sin x} = 1\right] \quad \textbf{(1)}$$

Since, function is continuous at $x = 0$

$$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \quad \frac{2}{a+1} = 2 \quad \textbf{(1/2)}$$

$$\Rightarrow \quad 2(a+1) = 2$$

$$\Rightarrow \quad a = 0$$

and $\quad b \,\varepsilon R, b \neq 0$ **($1\frac{1}{2}$)**

29. If $y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$, then prove that

$x(x+1)^2 y_2 + (x+1)^2 y_1 = 2.$ [4]

Ans :

We have, $$y = \log\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$$

$$= 2\log\left(\frac{x+1}{\sqrt{x}}\right)$$

$$= 2[\log(x+1) - \log\sqrt{x}]$$

$$= 2\left[\log(x+1) - \frac{1}{2}\log x\right]$$

$$\Rightarrow \quad y = 2\log(x+1) - \log x \quad \textbf{(1)}$$

On differentiating both sides w.r.t, x, we get

$$y_1 = \frac{2}{x+1} \cdot \frac{d}{dx}(x+1) - \frac{1}{x}$$

$$\Rightarrow \quad y_1 = \frac{2}{x+1} - \frac{1}{x}$$

$$\Rightarrow \quad y_1 = \frac{2x - (x+1)}{x(x+1)}$$

$$= \frac{x-1}{x(x+1)} \quad \text{...(i)} \quad \textbf{(1)}$$

Again, differentiating both sides w.r.t. x, we get

$$y_2 = \frac{x(x+1)\frac{d}{dx}(x-1)-(x-1)\frac{d}{dx}x(x+1)}{[x(x+1)]^2}$$

$$= \frac{(x^2+x)\times 1-(x-1)\left[x\frac{d}{dx}(x+1)+(x+1)\frac{d}{dx}x\right]}{x^2(x+1)^2}$$

$$= \frac{x^2+x-(x-1)[x\times 1+(x+1)\times 1]}{x^2(x+1)^2}$$

$$= \frac{x^2+x-(x-1)(2x+1)}{x^2(x+1)^2}$$

$$= \frac{x^2+x-(2x^2+x-2x-1)}{x^2(x+1)^2}$$

$$= \frac{x^2+x-(2x^2-x-1)}{x^2(x+1)^2}$$

$$= \frac{x^2+x-2x^2+x+1}{x^2(x+1)^2}$$

$\Rightarrow$ $$y_2 = \frac{-x^2+2x+1}{x^2(x+1)^2}$$ **(1)**

$\Rightarrow$ $$x(x+1)^2 y_2 = \frac{-x^2+2x+1}{x}$$

$$= \frac{2x-(x+1)(x-1)}{x}$$

$$= 2-\frac{(x+1)(x-1)}{x}$$

$\Rightarrow$ $$x(x+1)^2 y_2 = 2-(x+1)^2 y_1$$

[from Eq. (i)]

$$\Rightarrow x(x+1)^2 y_2 + (x+1)^2 y_1 = 2$$ **(1)**

Hence Proved

30. A person wants to plant some trees in his community park. The local nursery has to perform this task. It charges the cost of planting trees by the formula $C(x) = x^3 - 45x^2 + 600x$, where x is the number of trees and $C(x)$ is the cost of planting x tress in rupees. The local authority has imposed a restriction that it can plant 10 to 20 trees in one community park for a fair distribution. For how many trees should the person place the order so that he has to spend the least amount? How much is the least amount? Use calculus to answer these questions. [4]

Ans :

The cost of planting trees is given by

$$C(x) = x^3 - 45x^2 + 600x \quad ...(i)$$

where, $10 \le x \le 20$.

Let us assume that the function $C(x)$ is continuous in the interval [10, 20]. **(1)**

Now, on differentiating Eq. (i) w.r.t. x, we get

$$C'(x) = 3x^2 - 90x + 600$$
$$= 3(x^2 - 30x + 200)$$
$$= 3(x^2 - 20x - 10x + 200)$$
$$= 3[x(x-20) - 10(x-20)]$$

$\Rightarrow$ $$C'(x) = 3(x-20)(x-10)$$

We know that,

$$C'(x) = 0 \text{ at } x = 10$$

or $x = 20$ **(1)**

Thus, the maximum or the minimum value will occur at the points.

For maximum, put $x = 10$

$\therefore$ $$C(10) = (10)^3 - 45(10)^2 + 600(10)$$
$$= 1000 - 4500 + 6000$$
$$= 2500$$

For minimum, put $x = 20$

$\therefore$ $$C(20) = (20)^3 - 45(20)^2 + 600(20)$$
$$= 8000 - 18000 + 12000$$
$$= 2000$$

Hence, person must place the order for 20 trees and the least amount to be spent is equal to ₹2000. **(1)**

31. Find the equation (s) of the tangent (s) to the curve $y = (x^3-1)(x-2)$ at the points, where the curve intersects the X-axis. [4]

Ans :

Given equation of the curve is

$$y = (x^3-1)(x-2)$$

As the curve intersects with the X-axis.

Hence, coordinate of $y = 0$ at X-axis. **(1/2)**

When, $y = 0$

Equation of curve can be written as

$$y = (x-1)(x^2+x+1)(x-2)$$
$$0 = (x-1)(x^2+x+1)(x-2)$$

i.e. $x = 1$ or 2 **(1/2)**

On differentiating equation of curve w.r.t. x, we get

$$\frac{dy}{dx} = (3x^2)(x-2) + (x^3-1)\cdot 1$$
$$= 3x^6 - 6x^2 + x^3 - 1$$
$$= 4x^3 - 6x^2 - 1$$

Now, $$\left(\frac{dy}{dx}\right)_{(1,0)} = 4(1)^3 - 6(1)^2 - 1$$
$$= 4 - 6 - 1$$ **(1/2)**

$$= 4 - 7 = -3$$

and $$\left(\frac{dy}{dx}\right)_{(2,0)} = 4(2)^3 - 6(2)^2 - 1$$
$$= 4(8) - 6(4) - 1$$
$$= 32 - 24 - 1 = 7 \qquad \textbf{(1/2)}$$

Hence, the required equations of the tangents are

$$y - 0 = -3(x-1)$$
$$y = -3(x-1) = -3x + 3$$
$$\Rightarrow 3x + y = 3 \quad \textbf{(1)}$$

and $$y - 0 = 7(x-2) \Rightarrow y = 7x - 14$$

$$\Rightarrow \quad 7x - y = 14 \qquad \textbf{(1)}$$

or

Find the intervals in which the function:

$$f(x) = -3\log(1+x) + 4\log(2+x) - \frac{4}{2+x}$$

is strictly increasing or strictly decreasing.

Ans :

We have,

$$f(x) = -3\log(1+x) + 4\log(2+x) - \frac{4}{2+x}$$

(1)

On differentiating $f(x)$ w.r.t. x, we get

$$f'(x) = \frac{-3}{1+x} + \frac{4}{2+x} + \frac{4}{(2+x)^2}$$
$$= \frac{-3(2+x^2) + 4(1+x)(2+x) + 4(1+x)}{(1+x)(2+x)^2}$$
$$= \frac{-3(4+x^2+4x) + 4(2+2x+x+x^2) + 4 + 4x}{(1+x)(2+x)^2}$$
$$= \frac{-12 - 3x^2 - 12x + 4(2+3x+x^2) + 4 + 4x}{(1+x)(2+x)^2}$$
$$= \frac{-12 - 3x^2 - 12x + 8 + 12x + 4x^2 + 4 + 4x}{(1+x)(2+x)^2}$$
$$= \frac{x^2 + 4x}{(1+x)(2+x)^2} = \frac{x(x+4)}{(1+x)(2+x)^2} \qquad \textbf{(1)}$$

It is clear that domain of $f = (-1, \infty)$

Now Put, $f'(x) = 0$

i.e. $$\frac{x(x+4)}{(1+x)(2+x)^2} = 0$$

i.e. $$x = 0$$

$$[\because x \neq -4 \text{ as } -4\, \varepsilon(-1,\infty)] \quad \textbf{(1)}$$

In interval $(-1, 0)$, the expression

$$f'(x) = \frac{(-\text{ve})(+\text{ve})}{(+\text{ve})(+\text{ve})} = -\text{ ve}$$

Hence, in the interval $(-1, 0)$, f is strictly decreasing.

Now, in interval $(0, \infty)$, the expression

$$f'(x) = \frac{(+\text{ve})(+\text{ve})}{(+\text{ve})(+\text{ve})} = +\text{ ve}$$

Hence, in the interval $(0, \infty)$, f is strictly increasing. **(1)**

32. Find $\int \frac{\sec x}{1 + \operatorname{cosec} x} dx$. [4]

Ans :

Let $$I = \int \frac{\sec x\, dx}{1 + \operatorname{cosec} x} = \int \frac{\frac{1}{\cos x}}{1 + \frac{1}{\sin x}} dx$$
$$= \int \frac{\sin x\, dx}{\cos x(1 + \sin x)}$$
$$= \int \frac{\sin x \cos x\, dx}{\cos^2 x(1 + \sin x)} \qquad \textbf{(1/2)}$$
$$= \int \frac{\sin x \cos dx}{(1 - \sin^2 x)(1 + \sin x)}$$
$$= \int \frac{\sin x \cos x\, dx}{(1 - \sin x)(1 + \sin x)^2} \qquad \textbf{(1/2)}$$

$$[\because a^2 - b^2 = (a+b)(a-b)]$$

On putting $\sin x = t \Rightarrow \cos x\, dx = dt$, we get

$$I = \int \frac{t}{(1+t)^2(1-t)} dt$$

Now, $$\frac{t}{(1+t)^2(1-t)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{C}{1-t}$$

$$\Rightarrow t = A(1+t)(1-t) + B(1-t) + C(1+t)^2$$

Put $t = -1 \Rightarrow -1 = 2B$

$$\Rightarrow \quad B = \frac{-1}{2}$$

Put $t = 1$ **(1)**

$$\therefore \quad 1 = 4C \Rightarrow C = \frac{1}{4}$$

Put $t = 0$

$$\therefore \quad 0 = A + B + C$$
$$\Rightarrow \quad 0 = A + \left(\frac{-1}{2}\right) + \left(\frac{1}{4}\right)$$
$$\Rightarrow \quad A = \frac{1}{2} - \frac{1}{4}$$
$$= \frac{2-1}{4} = \frac{1}{4} \qquad \textbf{(1/2)}$$

Hence, the required integral

$$= \frac{1}{4}\int \frac{1}{1+t} dt + \left(\frac{-1}{2}\right)\int \frac{1}{(1+t)^2} dt + \frac{1}{4}\int \frac{1}{(1-t)} dt$$
$$= \frac{1}{4}\log|1+t| + \frac{-1}{2} \times \frac{-1}{1+t} - \frac{1}{4}\log|1-t| + C$$

(1/2)

$= \frac{1}{4}\log|1+\sin x| + \frac{1}{2(1+\sin x)} - \frac{1}{4}\log|1-\sin x|$

$+ C$ **(1)**

Section D

33. If the function $f: R \to R$ be defined by $f(x) = 2x - 3$ and $g: R \to R$ by $g(x) = x^3 + 5$, then find fog and show that fog is invertible. Also find $(\text{fog})^{-1}$ and hence find $fog^{-1}(9)$. [6]

Ans :

We have $f: R \to R$ defined by $f(x) = 2x - 3$ and $g: R \to R$ defined by $g(x) = x^3 + 5$.

Clearly, $fog: R \to R$ defined by

$$fog(x) = f(g(x))$$
$$= f(x^3+5) = 2(x^3+5) - 3$$
$$= 2x^3 + 7 \quad \textbf{(1)}$$

Now, let us show fog is invertible. For this, we shall show that fog is one-one and onto.

One-one Let $x_1, x_2 \varepsilon R(D_{fog})$ such that

$$fog(x_1) = fog(x_2)$$
$$\Rightarrow \quad 2x_1^3 + 7 = 2x_2^3 + 7$$
$$\Rightarrow \quad x_1^3 = x_2^3 \Rightarrow x_1^3 - x_2^3 = 0$$
$$\Rightarrow (x_1 - x_2)(x_1^2 + x_2^2 + x_1x_2) = 0$$
$$\Rightarrow \quad x_1 - x_2 = 0$$
$$\left[\because x_1^2 + 2x_1 \cdot \frac{x_2}{2} + \left(\frac{x_2}{2}\right)^2 + x_2^2 - \frac{x_2^2}{4} = (x_1 + x_2)^2 + \left(\frac{3x_2^2}{4}\right) > 0\right]$$
$$\Rightarrow \quad x_1 = x_2$$

Hence, fog is one-one. **(1)**

Onto Let $y \varepsilon R$ $(\text{Codomain})_{fog}$

Then, for any $x \Rightarrow fog(x) = y$

$$\Rightarrow \quad 2x^3 + 7 = y \Rightarrow 2x^3 = y - 7$$

i.e. $x = \sqrt[3]{\frac{y-7}{2}}, \varepsilon R(D_{fog})$ **(1)**

Thus, for every $y \varepsilon R(\text{Codomain})_{fog}$, there exist $\sqrt[3]{\frac{y-7}{2}} \varepsilon R(D_{fog})$ such that $fog\left(\sqrt[3]{\frac{y-7}{2}}\right) = y$.

Hence fog is onto. **(1)**

Since, fog is both one-one and onto, therefore it is invertible. **(1/2)**

Clearly, $(fog)^{-1}: R \to R$ defined by

$$(fog)^{-1}(x) = \sqrt[3]{\frac{x-7}{2}}$$

Now, $(fog)^{-1}(9) = \sqrt[3]{\frac{9-7}{2}}$

$= 1$ **($1\frac{1}{2}$)**

or

A binary operation $*$ is defined on the set R or real numbers by

$a * b = \begin{cases} a, & \text{if } b = 0 \\ |a| + b, & \text{if } b \neq 0 \end{cases}$. If at least one of a and b is 0, then prove that $a * b = b * a$. Check whether $*$ is commutative. Find the identity element for $*$, if it exists.

Ans :

Given, $a * b = \begin{cases} a, & \text{if } b = 0 \\ |a| + b, & \text{if } b \neq 0 \end{cases}$

Case I Let $a, b \varepsilon R$ such that $a = 0$ and $b \neq 0$

Then, $a * b = |a| + b = |0| + b = b$

and $b * a = b$

Thus, $a * b = b * a$ **(1)**

Case II Let $a, b \varepsilon R$ such that $a \neq 0$ and $b = 0$.

Then, $a * b = a$ and $b * a = |b| + a$

$= |0| + a = a$

Thus, $a * b = b * a$ **(1)**

Case III Let $a, b \varepsilon R$ such that $a = 0$ and $b = 0$

Then, $a * b = a = 0$ and $b * a = b = 0$.

Thus, $a * b = b * a$.

Hence, $a * b = b * a$, if at least one of a and b is 0. Now, we need to check whether * is commutative. For this, one more case is needed to be examined. **(1)**

Case IV Let $a, b \varepsilon R$ such that $a \neq 0$, $b \neq 0$.

Then, $a * b = |a| + b$ and $b * a = |b| + a$

Clearly, $a * b$ may not be equal to $b * a$

as $(-1) * 2 = 3, 2 * (-1) = 1$.

Hence, $(-1) * 2 \neq 2 * (-1)$

Thus, * is not commutative. **(1)**

Existence of identity: Let e be the identity element for *. Then $a * e = e * a = a, \forall a \varepsilon R$

Consider, $a * e = a$, this is possible only when $e = 0$.

Also, for $e = 0$ **(1)**

$$e * a = a = \begin{cases} 0, & \text{if } a = 0 \\ |0| + a, & \text{if } a \neq 0 \end{cases}$$

$$= \begin{cases} 0, & \text{if } a = 0 \\ a, & \text{if } a \neq 0 \end{cases}$$

$$= a$$

Thus, $e = 0 \varepsilon R$ such that $a * e = a = e * a$, $\forall a \varepsilon R$ Hence, 0 is the identity element. **(1)**

34. Using integration, find the area in the first quadrant bounded by the curve $y = x|x|$, the circle $x^2 + y^2 = 2$ and the Y-axis. [6]

Ans :

We have $y = x|x|$ and the equation of circle

$x^2 + y^2 = 2$

Now, $y = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$ **(1)**

But we take $y = x^2$, as it lies in first quadrant. Now, on solving $y = x^2$ and $x^2 + y^2 = 2$ simultaneously, we get

$$y + y^2 - 2 = 0$$

$$(y+2)(y-1) = 0$$

$$\Rightarrow \quad y = -2 \text{ or } y = 1$$

Since, $y = -2$ does not lie in 1st quadrant. **(1/2)**

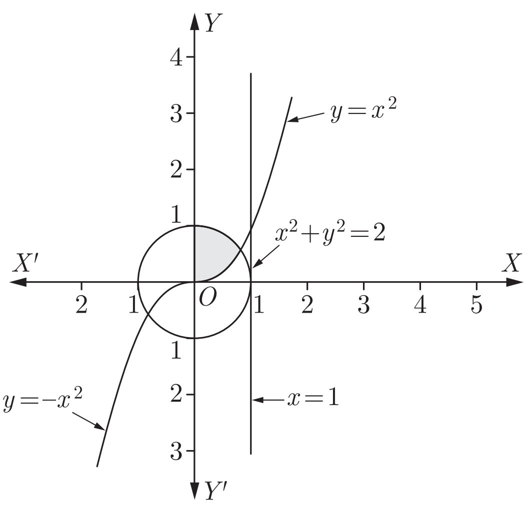
Thus, $y = 1$.

Put $y = 1$ in equation $y = x^2$.

$\therefore$ $x = 1$ [as $x = -1$ does not lie in 1st quadrant] **(1/2)**

Now, required area of the shaded region

$$= \int_0^1 (\sqrt{2 - x^2} - x^2)\, dx$$ **(1)**



(1)

$$= \frac{1}{2}\left[x\sqrt{2 - x^2} + 2\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1 - \frac{1}{3}[x^3]_0^1$$

$$= \frac{1}{2}\left[1\sqrt{2 - 1^2} + 2\sin^{-1}\frac{1}{\sqrt{2}}\right.$$

$$\left. - \left(0\sqrt{2 - 0^2} + 2\sin^{-1}\frac{0}{\sqrt{2}}\right)\right] - \frac{1}{3}(1^3 - 0^3)$$ **[1]**

$$= \frac{1}{2}\left[1 + 2\left(\frac{\pi}{4}\right)\right] - \frac{1}{3}(1)$$

$$= \frac{1}{2} + \frac{\pi}{4} - \frac{1}{3} = \frac{1}{2} - \frac{1}{3} + \frac{\pi}{4} = \frac{3-2}{6} + \frac{\pi}{4}$$

$$= \left(\frac{1}{6} + \frac{\pi}{4}\right) \text{ sq units.}$$ **(1)**

35. If $A = \begin{vmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{vmatrix}$, then find A^{-1} and hence solve the following system of equation. [6]

$$3x + 4y + 7z = 14,$$

$$2x - y + 3z = 4,$$

$$x + 2y - 3z = 0$$

Ans :

We have, $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ 7 & 3 & -3 \end{bmatrix}$

Then,

$|A| = 3(3-6) + (-2)(-12-14) + 1(1$

$= 62 \neq 0$

As, $|A| \neq 0$, then A^{-1} exists. **(1)**

Let C_{ij} represents the cofactor of (i,j)th element of A, then

$$C_{11} = (-1)^{1+1}(3-6) = -3$$

$$C_{12} = (-1)^{1+2}(-12-14) = 26$$

$$C_{13} = (-1)^{1+3}(12+7) = 19$$

$$C_{21} = (-1)^{2+1}(-6-3) = 9$$

$$C_{22} = (-1)^{2+2}(-9-7) = -16$$

$$C_{23} = (-1)^{2+3}(9-14) = 5$$

$$C_{31} = (-1)^{3+1}(4+1) = 5$$

$$C_{32} = (-1)^{3+2}(6-4) = -2$$

$$C_{33} = (-1)^{3+3}(-3-8) = -11$$ **(1)**

Now, $(\text{adj}A) = \begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{adj}A)$$

$$= \frac{1}{62}\begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 16 & 5 & -11 \end{bmatrix}$$

The given system of equation is equivalent to the matrix equation.

$A'X = B$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ 4 \\ 0 \end{bmatrix}$ **(1)**

$$\Rightarrow \quad X = (A')^{-1}B = (A^{-1})'B$$

$$\Rightarrow \quad X = \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}\begin{bmatrix} 14 \\ 4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{62}\begin{bmatrix} -42 & 104 & 0 \\ 126 & -64 & 0 \\ 70 & -8 & 0 \end{bmatrix}$$

$$= \frac{1}{62}\begin{bmatrix} 62 \\ 62 \\ 62 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ **(2)**

Hence, $x = 1$, $y = 1$ and $z = 1$ **(1)**

or

If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$, then find the inverse if A using elementary row transformations and hence solve the matrix equation $XA = [1,0,1]$.

Ans :

We have, $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$

We know that, $A = IA$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$ **(1/2)**

By applying $R_1 \leftrightarrow R_2$,

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}A$$ **(1)**

By applying $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 2 & 0 & 1 \end{bmatrix}A$$ **(1)**

By applying $R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -2 & 4 & 1 \end{bmatrix}A$$ **(1)**

By applying $R_2 \to R_2 + R_3$, $R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & 1 \\ -2 & 4 & 1 \end{bmatrix}A$$ **(1)**

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & 1 \\ -2 & 4 & 1 \end{bmatrix}$$ **(1/2)**

Now, $XA = [1\ 0\ 1]$

Post-multiplying by A^{-1}, we get

$$XAA^{-1} = [1\ 0\ 1]A^{-1}$$

or $X = [1\ 0\ 1]A^{-1}$

$$= [1\ 0\ 1]\begin{bmatrix} 2 & -3 & -1 \\ -1 & 2 & 1 \\ -2 & 4 & 1 \end{bmatrix}$$

$[\because AA^{-1} = I]$ **(1/2)**

$$= [2 + 0 - 2 - 3 + 0 + 4 - 1 + 0 + 1]$$

$X = [0\ 1\ 0]$ **(1/2)**

36. Find the distance of point $-2\hat{i} + 3\hat{j} - 4\hat{k}$ from the line $\vec{r} = \hat{i} + 2\hat{j} - \hat{k} + \lambda(\hat{i} + 3\hat{j} - 9\hat{k})$ measured parallel to the plane $x - y + 2z - 3 = 0$. [6]

Ans :

Given equation of line can be rewritten as

$$\vec{r} = (1 + \lambda)\hat{i} + (2 + 3\lambda)\hat{j} + (-1 - 9\lambda)\hat{k}$$

Thus, the general point on the given line is $(\lambda + 1, 3\lambda + 2, -9\lambda - 1)$ **(1)**

The direction ratios of the line parallel to the plane $x - y + 2z - 3 = 0$ intersecting the given line and passing through the point $(-2, 3, -4)$ are $(\lambda + 3, 3\lambda - 1, -9\lambda + 3)$ **(2)**

Since, line is parallel to the plane.

$$\therefore (\lambda + 3)1 + (3\lambda - 1)(-1) + (-9\lambda + 3)2 = 0$$

$$\Rightarrow \quad \lambda = \frac{1}{2}$$ **(1)**

Thus, the point of intersection of above line with the given line is $\left(\frac{3}{2}, \frac{7}{2}, -\frac{11}{2}\right)$

Now, required distance

$$= \sqrt{\left(\frac{3}{2} + 2\right)^2 + \left(\frac{7}{2} - 3\right)^2 + \left(\frac{-11}{2} + 4\right)^2}$$

$$= \frac{\sqrt{59}}{2} \text{ units}$$ **(2)**