Mathematics 2013 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. Write the principal value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ [1]

Solution : Given,

$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$$= \tan^{-1}\sqrt{3} - \left(\pi - \cot^{-1}\sqrt{3}\right)$$

$$[\because \cot^{-1}(-x) = \pi - \cot^{-1}x]$$

$$= \tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3} - \pi$$

$$= \frac{\pi}{2} - \pi = -\frac{\pi}{2}$$ **Ans.**

2. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$ [1]

Solution : $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$$= \tan^{-1}\left[2\sin\left\{2\cos^{-1}\left(\cos\frac{\pi}{6}\right)\right\}\right]$$

$$= \tan^{-1}\left[2\sin\left\{2\left(\frac{\pi}{6}\right)\right\}\right]$$

$$= \tan^{-1}\left[2\sin\left(\frac{\pi}{3}\right)\right]$$

$$= \tan^{-1}\left[2\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \tan^{-1}(\sqrt{3})$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{3}\right)\right]$$

$$= \frac{\pi}{3}.$$ **Ans.**

3. For what value of x, is the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew symmetric matrix ? [1]

Solution : We see that,

$$a_{31} = x$$

Given that the matrix 'A' is skew symmetric

$$\therefore \quad a_{ij} = -a_{ji}$$

$$\Rightarrow \quad a_{31} = -a_{13}$$

$$\therefore \quad x = -(-2) = 2.$$ **Ans.**

4. If matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $A^2 = KA$, then write the value of K. [1]

Solution : Given, $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

and $$A^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+(-1)\times(-1) & 1\times(-1)+(-1)\times1 \\ (-1)\times1+1\times(-1) & (-1)\times(-1)+1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow \quad A^2 = 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2A$$

Also,

$$A^2 = KA$$

$$\therefore \quad K = 2.$$ **Ans.**

5. Write the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant. [1]

Solution : We have,

$$y = mx$$

On differentiating, we get

$$\frac{dy}{dx} = m \quad \text{...(i)}$$

and $$m = \frac{y}{x} \quad \text{...(ii) (Given)}$$

From (i) and (ii), we get

$$\frac{dy}{dx} = \frac{y}{x}$$

The differential equation representing the family of curves $y = mx$ is

$$xdy - ydx = 0.$$ **Ans.**

6. If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ then write the value of a_{32}. A_{32}. [1]

Solution : Let $A = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

$$A_{32} = -\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = -(8-30) = 22$$

$$a_{32} = 5$$

Thus, $a_{32}.A_{32} = 5 \times 22 = 110.$ **Ans.**

7. P and Q are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$ respectively. Write the position vector of a point R which divides the line segment PQ in the ratio 2 : 1 externally. [1]

Solution : Position vector of point

$$R = \frac{2(\vec{a}+\vec{b})-1(3\vec{a}-2\vec{b})}{2-1}$$

$$= 2\vec{a} + 2\vec{b} - 3\vec{a} + 2\vec{b}$$

$$= -\vec{a} + 4\vec{b}.$$ **Ans.**

8. Find $|\vec{x}|$ if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 15$. [1]

Solution : We have,

$(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 15$

$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 15$

$\Rightarrow |\vec{x}|^2 = 15 + |\vec{a}|^2$ $(\because |\vec{a}| = 1)$

$\Rightarrow |\vec{x}| = \sqrt{15 + 1^2}$

$\therefore |\vec{x}| = 4.$ **Ans.**

9. Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$. [1]

Solution : Length of perpendicular

$$= \left|\frac{2(0)-3(0)+6(0)+21}{\sqrt{2^2+(-3)^2+6^2}}\right| = \frac{21}{\sqrt{49}}$$

$$= \frac{21}{7}$$

$$= 3.$$ **Ans.**

10. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupee) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$ and write which value does the question indicates. [1]

Solution : Total revenue, $R(x) = 3x^2 + 36x + 5$

Marginal revenue,

$$\frac{dR(x)}{dx} = 6x + 36$$

At $x = 5$,

$$\frac{dR(x)}{dx} = 6(5) + 36 = 66$$

Thus, marginal revenue = 66.

Money for welfare of employees is a nice step, there should be a growth in raising funds for the welfare of the employees. **Ans.**

SECTION — B

11. Consider $f : R^+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$ where R^+ is the set of all non-negative real numbers. [4]

Solution : Given, $f(x) = x^2 + 4$

Let $f(x_1) = f(x_2)$

$\Rightarrow x_1^2 + 4 = x_2^2 + 4$

$\Rightarrow x_1 = x_2$

Thus, $f(x)$ is one-one.

Since, $x^2 + 4$ is a real number. Thus, for every y in the co-domain of f, there exists a number x in R^+ such that

$$f(x) = y = x^2 + 4$$

Thus, we can say that $f(x)$ is onto.

Now, $f(x)$ is one-one and onto. Hence, $f(x)$ is invertible.

Let $f(x) = y \Rightarrow x^2 + 4 = y$

$\Rightarrow x^2 = y - 4$

i.e. $x = \sqrt{y-4}$

Also, $x = f^{-1}(y)$

$f^{-1}(y) = \sqrt{y-4}.$ **Hence Proved.**

12. Show that : $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$. [4]

Solution : Let $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = x$

$\Rightarrow \frac{1}{2}\sin^{-1}\frac{3}{4} = \tan^{-1}x$

$\sin^{-1}\frac{3}{4} = 2\tan^{-1}x$

$\Rightarrow \sin^{-1}\frac{3}{4} = \sin^{-1}\frac{2x}{1+x^2}$

$\left[\because \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x\right]$

Thus,

$$\frac{2x}{1+x^2} = \frac{3}{4}$$

$$\Rightarrow \quad 8x = 3+3x^2$$

$$\Rightarrow \quad 3x^2-8x+3 = 0$$

On comparing with

$ax^2+bx+c = 0$, we get

$$a = 3, b=-8, c=3$$

$$\therefore \quad x = \frac{8\pm\sqrt{64-36}}{6}$$

$$\left[\because x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$$

$$\Rightarrow \quad x = \frac{4\pm\sqrt{7}}{3}$$

But $\sin 2\theta = \frac{3}{4}$

$$\Rightarrow \quad 0<2\theta<\frac{\pi}{2}$$

$$\Rightarrow \quad 0<\theta<\frac{\pi}{4}$$

Accordingly,

$$0<\tan\theta<\tan\frac{\pi}{4}$$

$$0<\tan\theta<1$$

or $\quad 0<x<1$

Thus $\quad x = \frac{4+\sqrt{7}}{3}$ is rejected

$$\Rightarrow \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3} = \text{R.H.S.}$$

Hence Proved.

OR

Solve the following equation :

$$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$

Solution : Given,

$$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$

$$\Rightarrow \quad \cos\left(\tan^{-1}\frac{x}{1}\right) = \sin\left(\tan^{-1}\frac{4}{3}\right)$$

$$\Rightarrow \quad \cos\left(\tan^{-1}\frac{x}{1}\right) = \cos\left(\frac{\pi}{2}-\left(\tan^{-1}\frac{4}{3}\right)\right)$$

On comparing

$$\tan^{-1}\frac{x}{1} = \frac{\pi}{2}-\tan^{-1}\frac{4}{3}$$

$$\Rightarrow \tan^{-1}x+\tan^{-1}\frac{4}{3}=\frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x+\frac{4}{3}}{1-\frac{4}{3}x}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{3x+4}{3}}{\frac{3-4x}{3}}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{3x+4}{3-4x} = \tan\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \frac{3x+4}{3-4x} = \infty$$

$$\Rightarrow \quad \frac{3-4x}{3x+4} = 0$$

$$\Rightarrow \quad 3-4x=0$$

$$\therefore \quad x = \frac{3}{4}$$

Ans.

13. Using properties of determinants prove the following :

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$$ **[4]**

Solution : L.H.S.

$$= \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Applying $R_1 \to R_1+R_2+R_3$, we get

$$= \begin{vmatrix} 3x+3y & 3x+3y & 3x+3y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Taking $(3x+3y)$ common from R_1, we get

$$= (3x+3y)\begin{vmatrix} 1 & 1 & 1 \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Applying $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1$,

$$= 3(x+y)\begin{vmatrix} 1 & 0 & 0 \\ x+2y & -2y & -y \\ x+y & y & -y \end{vmatrix}$$

Taking $-y$ and y common from C_3 and C_2 respectively,

$$= -3y^2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ x+2y & -2 & 1 \\ x+y & 1 & 1 \end{vmatrix}$$

Expanding along R_1, we get

$$= -3y^2(x+y)\,[1.(-2-1)]$$
$$= 9y^2(x+y) = \text{R.H.S.}$$

Hence Proved

14. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$. **[4]**

Solution : Given,

$$y^x = e^{y-x}$$

Taking log on both sides,

$$x \log y = (y-x)\log e$$
$$\Rightarrow \quad x\log y = y - x \qquad (\because \log e = 1)$$
$$\Rightarrow \quad y = x\log y + x \qquad \text{...(i)}$$

On differentiating, we get

$$\frac{dy}{dx} = x.\frac{1}{y}.\frac{dy}{dx} + \log y + 1$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{y}.\frac{dy}{dx} + \log y + 1$$
$$\Rightarrow \quad \left(1 - \frac{x}{y}\right)\frac{dy}{dx} = 1 + \log y$$
$$\Rightarrow \quad \left(\frac{y-x}{y}\right)\frac{dy}{dx} = 1 + \log y$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(1+\log y)}{y-x}$$

Put the value of y from equation (i),

$$\frac{dy}{dx} = \frac{(x\log y + x)\,(1+\log y)}{x\log y + x - x}$$
$$\frac{dy}{dx} = \frac{(x\log y + x)(1+\log y)}{x\log y}$$
$$\therefore \quad \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}.$$

Hence Proved.

15. Differentiate the following with respect to x :

$$\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right). \qquad \textbf{[4]}$$

Solution :

Let,
$$y = \sin^{-1}\left[\frac{2^{x+1}.3^x}{1+(36)^x}\right]$$
$$= \sin^{-1}\left[\frac{2.2^x.3^x}{1+(6\times 6)^x}\right]$$
$$= \sin^{-1}\left[\frac{2(6^x)}{1+(6^x)^2}\right]$$
$$\therefore \quad y = 2\tan^{-1}(6^x)$$
$$\left[\because \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1} x\right]$$

On differentiating, we get

$$\frac{dy}{dx} = \frac{2}{1+(6^x)^2} \times [(6^x)\log 6]$$
$$\frac{dy}{dx} = \frac{2(6^x)\log 6}{1+(36)^x} = \frac{2.2^x.3^x \log 6}{1+(36)^x}$$
$$\frac{dy}{dx} = \left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)\log 6. \qquad \textbf{Ans.}$$

16. Find the value of k, for which

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

continuous at $x = 0$. **[4]**

Solution : At $x = 0$,

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0}\frac{\left(\sqrt{1-kh}-\sqrt{1+kh}\right)}{-h} \times \frac{\left(\sqrt{1-kh}+\sqrt{1+kh}\right)}{\left(\sqrt{1-kh}+\sqrt{1+kh}\right)}$$
$$= \lim_{h\to 0}\frac{(1-kh-1-kh)}{-h\left(\sqrt{1-kh}+\sqrt{1+kh}\right)}$$
$$= \frac{2k}{\sqrt{1-0}+\sqrt{1-0}} = \frac{2k}{2} = k$$
$$\text{R.H.L} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0}\frac{2h+1}{h-1} = \frac{0+1}{0-1} = -1$$

Since, $f(x)$ is continuous,

$$\therefore \quad \text{L.H.L} = \text{R.H.L}$$
$$\Rightarrow \quad k = -1. \qquad \textbf{Ans.}$$

OR

If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$.

Solution :

Given, $\quad x = a\cos^3\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad ...(i)$$

and $\quad y = a\sin^3\theta$

$$\Rightarrow \quad \frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\tan\theta$$

and $\quad \frac{d^2y}{dx^2} = -\sec^2\theta.\frac{d\theta}{dx}$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\sec^2\theta}{3a\cos^2\theta\sin\theta} \quad \text{[using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta\sin\theta}$$

$$\left(\frac{d^2y}{dx^2}\right)_{at\,\theta=\frac{\pi}{6}} = \frac{1}{3a\sin\frac{\pi}{6}\cos^4\frac{\pi}{6}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{3a\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^4}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{32}{27a} \quad \textbf{Ans.}$$

17. Evaluate : $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx$. **[4]**

Solution : $\frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}$

$$= \frac{-2\sin\left(\frac{2x+2\alpha}{2}\right)\sin\left(\frac{2x-2\alpha}{2}\right)}{-2\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \frac{\left[\left\{2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right\}\left\{2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right\}\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

$$= 4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$= 2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\left(\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right)\right]$$

$[\because \cos(A+B)+\cos(A-B) = 2\cos A\cos B]$

$$= 2\cos x + 2\cos\alpha$$

Now,

$$\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha} = \int(2\cos x+2\cos\alpha)dx$$

$$= 2\sin x + 2x\cos\alpha + C. \quad \textbf{Ans.}$$

OR

Evaluate : $\int\frac{x+2}{\sqrt{x^2+2x+3}}dx$

Solution : $\int\frac{x+2}{\sqrt{x^2+2x+3}}dx$

Let, $\quad x+2 = A\frac{d}{dx}(x^2+2x+3)+B$

$$\Rightarrow \quad x+2 = A(2x+2)+B$$

$$\Rightarrow \quad x+2 = 2Ax+2A+B$$

On equating the coefficient of x and constant term on both sides, we get

$$1 = 2A$$

$$\Rightarrow \quad A = \frac{1}{2}$$

and $\quad 2 = 2A + B$

$$\Rightarrow \quad 2 = 2\times\frac{1}{2}+B \Rightarrow B = 2-1 = 1$$

$$\therefore \quad \int\frac{x+2}{\sqrt{x^2+2x+3}}dx$$

$$= \int\frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int\frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx + \int\frac{1}{\sqrt{x^2+2x+3}}dx$$

$$= I_1 + I_2 \quad ...(i)$$

Now,

$$I_1 = \frac{1}{2}\int\frac{2x+2}{\sqrt{x^2+2x+3}}dx$$

Let $\quad x^2+2x+3 = t$

$$\Rightarrow \quad (2x+2)\,dx = dt$$

$$\therefore \quad I_1 = \frac{1}{2}\int\frac{1}{\sqrt{t}}dt = \frac{1}{2}\int t^{-1/2}dt = \frac{1}{2}.\frac{t^{-1/2+1}}{-1/2+1}+C_1$$

$$= \frac{1}{2}(2\sqrt{t})+C_1$$

$$= \sqrt{x^2+2x+3}+C_1 \quad ...(ii)$$

and

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$= \int \frac{1}{\sqrt{x^2+2x+1+2}}dx$$

$$= \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}dx$$

$$= \log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C_2 \quad ...(iii)$$

Put the value of I_1 and I_2 in equation (i), we get

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \sqrt{x^2+2x+3} + \log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$$

(where $C = C_1 + C_2$) **Ans.**

18. Evaluate $\int \frac{dx}{x(x^5+3)}$. **[4]**

Solution : $\int \frac{dx}{x(x^5+3)} = \int \frac{x^4}{x^5(x^5+3)}dx$

Put, $(x^5+3) = t$

$\Rightarrow \quad 5x^4 dx = dt$

$\Rightarrow \quad x^4\, dx = \frac{dt}{5}$

$\therefore \quad \int \frac{dx}{x(x^5+3)} = \frac{1}{5}\int \frac{dt}{t(t-3)}$

Let $\quad \frac{1}{t(t-3)} = \frac{A}{t}+\frac{B}{t-3}$

$1 = A(t-3) + Bt$

$\Rightarrow \quad 1 = At - 3A + Bt$

$\Rightarrow \quad 1 = (A+B)\,t - 3A$

On comparing the co-efficient, we get

$1 = -3A$

$\Rightarrow \quad A = -1/3$

$A + B = 0$

$\Rightarrow \quad B = 1/3$

Now,

$$\frac{1}{5}\int \frac{dt}{t(t-3)} = \frac{1}{5}\int\left(\frac{-1}{3t}+\frac{1}{3(t-3)}\right)dt$$

$$= \frac{-1}{15}\log t+\frac{1}{15}\log(t-3)+C$$

$$= \frac{-1}{15}\left(\log\left(\frac{t}{t-3}\right)\right)+C$$

$[\because \log(a/b) = \log a - \log b]$

$$= \frac{-1}{15}\left(\log\frac{(x^5+3)}{(x^5+3-3)}\right)+C$$

$$= \frac{1}{15}\left(\log\frac{x^5}{x^5+3}\right)+C. \quad \textbf{Ans.}$$

19. Evaluate : $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}dx$. **[4]**

Solution : Let, $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}dx$

We know that,

$$\int_0^{2a} f(x)dx = \int_0^{a}\{f(x)+f(2a-x)\}dx$$

$$\therefore \quad I = \int_0^{\pi}\left(\frac{1}{1+e^{\sin x}}+\frac{1}{1+e^{\sin(2\pi-x)}}\right)dx$$

$$= \int_0^{\pi}\left(\frac{1}{1+e^{\sin x}}+\frac{1}{1+e^{-\sin x}}\right)dx$$

$$= \int_0^{\pi}\left(\frac{1}{1+e^{\sin x}}+\frac{e^{\sin x}}{1+e^{\sin x}}\right)dx$$

$$= \int_0^{\pi}\left(\frac{1+e^{\sin x}}{1+e^{\sin x}}\right)dx$$

$$= \int_0^{\pi} dx = [x]_0^{\pi}$$

$$= \pi. \quad \textbf{Ans.}$$

20. If $\vec{a} = \hat{i}-\hat{j}+7\hat{k}$ **and** $\vec{b} = 5\hat{i}-\hat{j}+\lambda\hat{k}$, **then find the value of** λ, **so that** $\vec{a}+\vec{b}$ **and** $\vec{a}-\vec{b}$ **are perpendicular vectors.** **[4]**

Solution : Given $\vec{a} = \hat{i}-\hat{j}+7\hat{k}$

and $\quad \vec{b} = 5\hat{i}-\hat{j}+\lambda\hat{k}$

$\therefore \quad \vec{a}+\vec{b} = \hat{i}-\hat{j}+7\hat{k}+5\hat{i}-\hat{j}+\lambda\hat{k}$

$= 6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}$

and $\quad \vec{a}-\vec{b} = \hat{i}-\hat{j}+7\hat{k}-5\hat{i}+\hat{j}-\lambda\hat{k}$

$= -4\hat{i}+(7-\lambda)\hat{k}$

Since $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular vectors.

$\therefore \quad (\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 0$

$[6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}]\,[-4\hat{i}+(7-\lambda)\hat{k}] = 0$

$\Rightarrow -24+(7+\lambda)(7-\lambda) = 0$

$\Rightarrow \quad -24+49-\lambda^2 = 0$

$\Rightarrow \quad \lambda^2 = 25$

$\therefore \quad \lambda = \pm 5.$ **Ans.**

21. Show that the lines

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$$
$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$$

are intersecting. Hence find their point of intersection. [4]

Solution : Consider,

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}) \quad ...(i)$$

$\therefore \quad \vec{r} = (3+\lambda)\hat{i} + (2+2\lambda)\hat{j} + (-4+2\lambda)\hat{k}$

Put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in above equation

$\therefore x\hat{i} + y\hat{j} + z\hat{k} = (3+\lambda)\hat{i} + (2+2\lambda)\hat{j} + (-4+2\lambda)\hat{k}$

Equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$x = 3 + \lambda, y = 2 + 2\lambda, z = -4 + 2\lambda$

$\therefore$ Coordinates of any point on line (i) are

$(3 + \lambda, 2 + 2\lambda, -4 + 2\lambda)$

Consider,

$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k}) \quad ...(ii)$$

$\therefore \quad \vec{r} = (5+3\mu)\hat{i} + (-2+2\mu)\hat{j} + 6\mu\hat{k}$

Put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in the above equation, we get

$x\hat{i} + y\hat{j} + z\hat{k} = (5+3\mu)\hat{i} + (-2+2\mu)\hat{j} + 6\mu\hat{k}$

Equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$x = 5 + 3\mu, y = -2 + 2\mu, z = 6\mu$

$\therefore$ Coordinates of any point on line (ii) are

$(5 + 3\mu, -2 + 2\mu, 6\mu)$

Line (i) and (ii) intersect if

$3 + \lambda = 5 + 3\mu \Rightarrow \lambda - 3\mu = 2 \quad ...(iii)$

$2 + 2\lambda = -2 + 2\mu \Rightarrow 2\lambda - 2\mu = -4 \quad ...(iv)$

$-4 + 2\lambda = 6\mu \Rightarrow 2\lambda - 6\mu = 4 \quad ...(v)$

Subtracting equations (iv) from (v), we get

$4\mu = -8$

$\Rightarrow \quad \mu = -2$

From equation (iv),

$2\lambda - 2(-2) = -4$

$\Rightarrow \quad 2\lambda = -4 - 4$

$\Rightarrow \quad \lambda = \frac{-8}{2} = -4$

Put the value of λ and μ in equation (iii),

$\therefore \quad \lambda - 3\mu = -4 - 3(-2)$

$= -4 + 6$

$= 2$

Putting the value of λ in $(3 + \lambda, 2 + 2\lambda, -4 + 2\lambda)$, we get point $(-1, -6, -12)$

$\therefore$ Point of intersection is $(-1, -6, -12)$. **Ans.**

OR

Find the vector equation of the plane through the points (2, 1, –1) and (–1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$.

Solution : Equation of plane passing through (2, 1, –1) is

$$a(x-2) + b(y-1) + c(z+1) = 0 \quad ...(i)$$

This plane passes through (–1, 3, 4)

Thus, $a(-1-2) + b(3-1) + c(4+1) = 0$

$\Rightarrow \quad -3a + 2b + 5c = 0 \quad ...(ii)$

Also, the above plane is perpendicular to the plane

$x - 2y + 4z = 10$

$\therefore \quad a(1) + b(-2) + c(4) = 0$

$\Rightarrow \quad a - 2b + 4c = 0 \quad ...(iii)$

Now, we have

$-3a + 2b + 5c = 0$

and $\quad a - 2b + 4c = 0$

Solving the above equation by cross multiplication, we get

$$\frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-2}$$

$$\Rightarrow \quad \frac{a}{18} = \frac{b}{17} = \frac{c}{4} = \lambda$$

$\Rightarrow \quad a = 18\lambda, \quad b = 17\lambda, \quad c = 4\lambda$

$\therefore$ Eq. (i) becomes

$18\lambda(x-2) + 17\lambda(y-1) + 4\lambda(z+1) = 0$

$\Rightarrow \quad 18x + 17y + 4z - 36 - 17 + 4 = 0$

$\Rightarrow \quad 18x + 17y + 4z - 49 = 0$

which is cartesian equation of the plane.

$\therefore$ The required vector equation of plane is

$\vec{r}.(18\hat{i} + 17\hat{j} + 4\hat{k}) = 49.$ **Ans.**

22. The probabilities of two students A and B coming to the school in time are $\frac{3}{7}$ and $\frac{5}{7}$ respectively. Assuming that the events, A coming in time and B coming in time are independent. Find the probability of only one of them coming to the school in time. Write atleast one advantages of coming to school in time. [4]

Solution : Let probability that A comes on time

$$= P(A)$$

Let probability that B comes on time

$$= P(B)$$

$$\therefore \quad P(A) = \frac{3}{7}$$

$$P(B) = \frac{5}{7}$$

Probability of only one of them coming to school on time

$$= P(A\,\bar{B} \text{ or } \bar{A}\,B)$$

$$= P(A)\,P(\bar{B}) + P(\bar{A})\,P(B)$$

$$= P(A)\,[1 - P(B)] + P(B)\,[1 - P(A)]$$

$$= \frac{3}{7}\left(1 - \frac{5}{7}\right) + \frac{5}{7}\left(1 - \frac{3}{7}\right)$$

$$= \frac{3}{7}\cdot\frac{2}{7} + \frac{5}{7}\cdot\frac{4}{7}$$

$$= \frac{6}{49} + \frac{20}{49} = \frac{26}{49}.$$

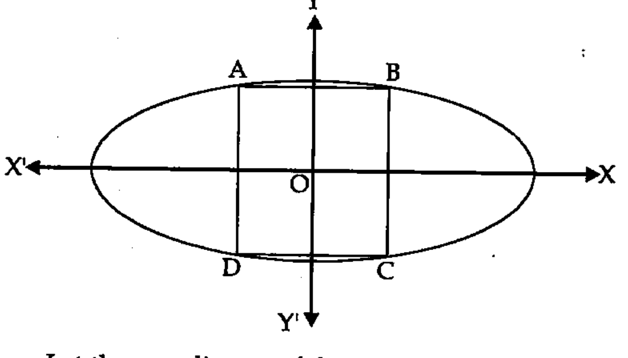
Student will not get punishment if he reach on time. **Ans.**

SECTION — C

23. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **[6]**

Solution : Equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$



Let the coordinates of the points be

$$A = (-a\cos\theta, b\sin\theta)$$

$$B = (a\cos\theta, b\sin\theta)$$

$$C = (a\cos\theta, -b\sin\theta)$$

and $\quad D = (-a\cos\theta, -b\sin\theta)$

Thus,

$$AB = 2a\cos\theta$$

$$BC = 2b\sin\theta$$

Area of rectangle ABCD,

$$S = AB \times BC$$

$$= (2a\cos\theta).(2b\sin\theta)$$

$$\Rightarrow \quad S = 2ab\sin 2\theta$$

On differentiating, we get

$$\frac{dS}{d\theta} = 2ab(2\cos 2\theta)$$

$$\Rightarrow \quad \frac{dS}{d\theta} = 4ab\cos 2\theta \qquad \text{...(ii)}$$

For greatest area,

$$\therefore \quad \frac{dS}{d\theta} = 0$$

$$\Rightarrow \quad 4ab\cos 2\theta = 0$$

$$\Rightarrow \quad \cos 2\theta = 0$$

$$\Rightarrow \quad \cos 2\theta = \cos \pi/2$$

$$\Rightarrow \quad 2\theta = \pi/2$$

$$\Rightarrow \quad \theta = \frac{\pi}{4}$$

On further differentiation of equation (ii), we get

$$\frac{d^2S}{d\theta^2} = -8ab\sin 2\theta$$

Put $\theta = \frac{\pi}{4}$

$$\therefore \quad \left(\frac{d^2S}{d\theta^2}\right)_{\theta = \frac{\pi}{4}} = -8ab < 0$$

$\therefore$ Area is maximum when $\theta = \frac{\pi}{4}$

$S_{max} = 2ab\sin 2\left(\frac{\pi}{4}\right) = 2ab$ sq. units. **Ans.**

OR

Find the equations of tangents to the curve $3x^2 - y^2 = 8$ which pass through the point $\left(\frac{4}{3}, 0\right)$.

Solution : The equation of the curve is $3x^2 - y^2 = 8$.

On differentiating, we get

$$6x - 2y.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x}{y}$$

Let (x_1, y_1) be the point on the curve at which tangent passes through the point $\left(\frac{4}{3}, 0\right)$.

$\therefore \quad 3x_1^2 - y_1^2 = 8$

$y_1^2 = 3x_1^2 - 8 \quad ...(i)$

Slope of the tangent

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1}{y_1}$$

Equation of the tangent passing through the point (x_1, y_1) with slope $\frac{3x_1}{y_1}$ is

$$y - y_1 = \frac{3x_1}{y_1}(x - x_1)$$

Point $\left(\frac{4}{3}, 0\right)$ lies on the above tangent,

$$\therefore \quad 0 - y_1 = \frac{3x_1}{y_1}\left(\frac{4}{3} - x_1\right)$$

$\Rightarrow y_1^2 - 3x_1^2 + 4x_1 = 0$

$\Rightarrow \quad y_1^2 = 3x_1^2 - 4x_1 \quad ...(ii)$

Now, take equation (i) and (ii), we get

$3x_1^2 - 8 = 3x_1^2 - 4x_1$

$\Rightarrow \quad 4x_1 = 8$

$\Rightarrow \quad x_1 = 2$

From equation (ii), we get

$y_1^2 = 3(2)^2 - 4(2)$

$\Rightarrow \quad y_1^2 = 12 - 8$

$\Rightarrow \quad y_1^2 = 4$

$\Rightarrow \quad y_1 = \pm 2$

Now, equation of tangent at the point (2, 2) is given by,

$$y - 2 = \frac{3 \times 2}{2}(x - 2)$$

$\Rightarrow \quad y - 2 = 3x - 6$

$\Rightarrow \quad y - 3x + 4 = 0$

and equation of tangent at the point (2, – 2) is given by,

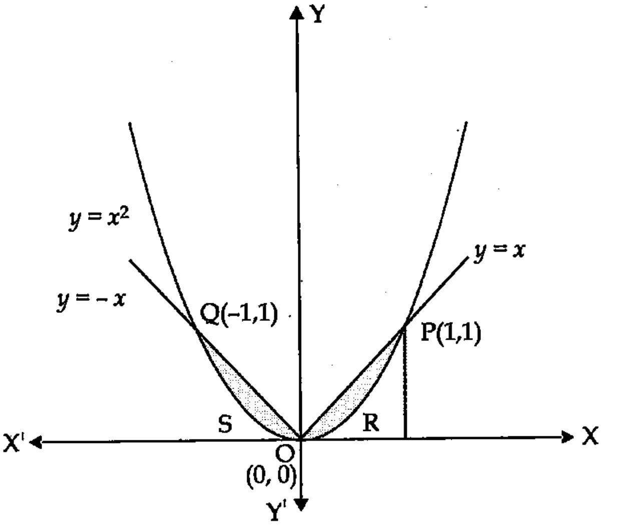
$$y + 2 = \frac{3 \times 2}{(-2)}(x - 2)$$

$\Rightarrow \quad y + 2 = -3x + 6$

$\Rightarrow \quad y + 3x - 4 = 0.$ **Ans.**

24. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$. **[6]**

Solution: Given parabola $y = x^2$ which is symmetrical about Y-axis and passes through O(0, 0) and the curve $y = |x|$. The point of intersection of parabola, $y = x^2$ and line, $y = x$ in the first quadrant is P(1, 1). The given area is symmetrical about Y–axis.



$\therefore$ Area of OPRO = Area of OQSO.

$\therefore$ Required area = 2 (Area of shaded region in the first quadrant)

$$= 2\int_0^1 (x - x^2)dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 2 \times \frac{1}{6}$$

$= \frac{1}{3}$ sq. units. **Ans.**

25. Find the particular solution of the differential equation $(\tan^{-1} y - x)dy = (1 + y^2)dx$, given that when $x = 0, y = 0$. **[6]**

Solution : The given differential equation is,

$$(\tan^{-1} y - x)dy = (1 + y^2)dx,$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{(\tan^{-1} y - x)}{(1 + y^2)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\tan^{-1} y}{1 + y^2} - \frac{x}{1 + y^2}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$$

On comparing with the form $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{1}{1 + y^2} \text{ and } Q = \frac{\tan^{-1} y}{1 + y^2}$$

Integrating factor (I.F.) $= e^{\int P dy}$

$$= e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$$

The solution is

$$x(\text{I.F.}) = \int Q.(\text{I.F.})dy$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = \int e^{\tan^{-1} y}.\frac{\tan^{-1} y}{1+y^2}dy$$

Let $t = \tan^{-1} y$ for R.H.S

$$\Rightarrow \quad \frac{dt}{dy} = \frac{1}{1+y^2}$$

$$\Rightarrow \quad dt = \frac{dy}{1+y^2}$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = \int e^t.t\,dt$$

Integrating by parts, we get

$$x.e^{\tan^{-1} y} = t.\int e^t dt - \int\left[\frac{d}{dt}t.\int e^t dt\right]dt$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = t(e^t) - \int e^t\,dt$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = te^t - e^t + C$$

$$\Rightarrow \quad x.e^{\tan^{-1} y} = \tan^{-1} y\, e^{\tan^{-1} y} - e^{\tan^{-1} y} + C$$

When $x = 0, y = 0$

$$0 = \tan^{-1} 0\left(e^{\tan^{-1} 0}\right) - e^{\tan^{-1} 0} + C$$

$$\Rightarrow \quad 0 = 0.\,e^0 - e^0 + C$$

$$\Rightarrow \quad 0 = 0 - 1 + C$$

$$\Rightarrow \quad C = 1$$

Thus,

$$x.e^{\tan^{-1} y} = \tan^{-1} y\left(e^{\tan^{-1} y}\right) - e^{\tan^{-1} y} + 1$$

$$\Rightarrow \quad x = \tan^{-1} y - 1 + e^{-\tan^{-1} y}.$$ **Ans.**

26. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i}+3\hat{j})-6=0$ and $\vec{r}.(3\hat{i}-\hat{j}-4\hat{k})=0$ whose perpendicular distance from origin is unity. [6]

Solution : The equation of the plane passes through the intersection of given planes is

I[st] plane + λ (2[nd] plane) = 0

$$[\vec{r}\,(\hat{i}+3\hat{j})-6]+\lambda[\vec{r}.(3\hat{i}-\hat{j}-4\hat{k})]=0$$

$$\Rightarrow \quad \vec{r}\,[(1+3\lambda)\hat{i}+(3-\lambda)\hat{j}-4\lambda\hat{k}]-6=0 \quad ...(i)$$

Length of perpendicular from the origin is

$$\left|\frac{-6}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}\right| = 1$$

$$\Rightarrow \quad (1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2 = 36$$

$$\Rightarrow \quad 1+9\lambda^2+6\lambda+9+\lambda^2-6\lambda+16\lambda^2 = 36$$

$$\Rightarrow \quad 26\lambda^2 = 26$$

$$\Rightarrow \quad \lambda^2 = 1$$

$$\therefore \quad \lambda = \pm 1$$

Put the value of 'λ' in equation (i), we get

When, $\lambda = 1$

$$\vec{r}.[(1+3)\hat{i}+(3-1)\hat{j}-4\hat{k}]-6 = 0$$

$$\vec{r}.(4\hat{i}+2\hat{j}-4\hat{k})-6 = 0$$

$$\vec{r}.(2\hat{i}+\hat{j}-2\hat{k}) = 3$$

When, $\lambda = -1$

$$\vec{r}.[(1-3)\hat{i}+(3+1)\hat{j}+4\hat{k}]-6 = 0$$

$$\Rightarrow \quad \vec{r}.(-2\hat{i}+4\hat{j}+4\hat{k})-6 = 0$$

$$\Rightarrow \quad \vec{r}.(-\hat{i}+2\hat{j}+2\hat{k}) = 3$$

Hence, equations of the planes are $\vec{r}.(2\hat{i}+\hat{j}-2\hat{k}) = 3$ and $\vec{r}.(-\hat{i}+2\hat{j}+2\hat{k})=3.$ **Ans.**

OR

Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes $\vec{r}.(\hat{i}-\hat{j}+2\hat{k})=5$ and $\vec{r}.(3\hat{i}-\hat{j}+\hat{k})=6$.

Solution : The equation of line passing through the point (1, 2, 3) is

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z-3}{c} \quad ...(i)$$

The given planes are

$$\vec{r}.(\hat{i}-\hat{j}+2\hat{k})=5$$

$$\therefore \quad x-y+2z=5 \quad ...(ii)$$

and $\vec{r}.(3\hat{i}+\hat{j}+\hat{k})=6$

$$\therefore \quad 3x+y+z=6 \quad ...(iii)$$

Since line (i) is parallel to both planes (ii) and (iii),

∴ It is perpendicular to the normal to the planes

$$a.1+b(-1)+c.2=0$$

$$\Rightarrow \quad a-b+2c=0$$

and $a.3+b.1+c.1=0$

$$\Rightarrow \quad 3a+b+c=0$$

Solving these equations, we get

$$\frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3}$$

$$\Rightarrow \quad \frac{a}{-3} = \frac{b}{5} = \frac{c}{4}$$

∴ From (i), the required line is

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

∴ The vector equation of the given line is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

$$\Rightarrow \quad \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 4\hat{k}).$$ **Ans.**

27. In a hockey match, both teams A and B scored same number of goals up to the end of the game, so to decide the winner, the referee asked both the captains to throw a die alternately and decided that the team, whose captain gets a six first, will be declared the winner. If the captain of team A was asked to start, find their respective probabilities of winning the match and state whether the decision of the referee was fair or not. [6]

Solution : Probability of getting a six by the captains of both the teams A and B is

$$P(A) = \frac{1}{6}$$

and $$P(B) = \frac{1}{6}$$

$$\therefore \quad P(\bar{A}) = P(\bar{B}) = 1 - \frac{1}{6} = \frac{5}{6}$$

Since A starts the game, he can throw a six in the following mutually exclusive ways :

$(A), (\bar{A}\bar{B}A), (\bar{A}\bar{B}\bar{A}\bar{B}A), \ldots$

By the theorem of Total Probability,

∴ Probability that A wins

$$= P(A) + P(\bar{A}\bar{B}A) + P(\bar{A}\bar{B}\bar{A}\bar{B}A) + \ldots$$

$$= P(A) + P(\bar{A})P(\bar{B})P(A) + P(\bar{A})P(\bar{B})P(\bar{A})P(\bar{B})P(A) + \ldots$$

$$= \frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \ldots$$

$$= \frac{1}{6} + \frac{1}{6}\cdot\left(\frac{5}{6}\right)^2 + \frac{1}{6}\cdot\left(\frac{5}{6}\right)^4 + \ldots$$

This is an infinite G.P.

$$\therefore \quad a = \frac{1}{6}$$

and $$r = \left(\frac{5}{6}\right)^2$$

Hence, the probability of the team A winning the match

$$= \frac{\frac{1}{6}}{1 - \left(\frac{5}{6}\right)^2} \quad \left(\text{Using } S_\infty = \frac{a}{1-r}\right)$$

$$= \frac{\frac{1}{6}}{\frac{11}{36}} = \frac{6}{11}$$

Since, the total probability is unity, the probability of team B winning the match $1 - \frac{6}{11} = \frac{5}{11}$.

The decision of the referee was not fair as whosoever starts throwing the die gets an upper hand. **Ans.**

28. A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital respectively; which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3 workers and 1 unit of capital is required to produce one unit of B. If A and B are priced at ₹ 100 and ₹ 120 per unit respectively, how should he use his resources to maximize the total revenue ? Form the above as an LPP and solve graphically.

Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at the same rate? [6]

Solution : Let x units of the goods A and y units of goods B be produced to maximize the total revenue. Then the total revenue is $Z = 100x + 120y$. This is a linear function which is to be maximized. Hence it is the objective function. The constraints are as per the following table :

	Unit A	Unit B	Total Units
Workers	2	3	30
Capital	3	1	17

From the table, the constraints are

$$2x + 3y \le 30$$
$$3x + y \le 17$$

and $$x \ge 0, y \ge 0$$

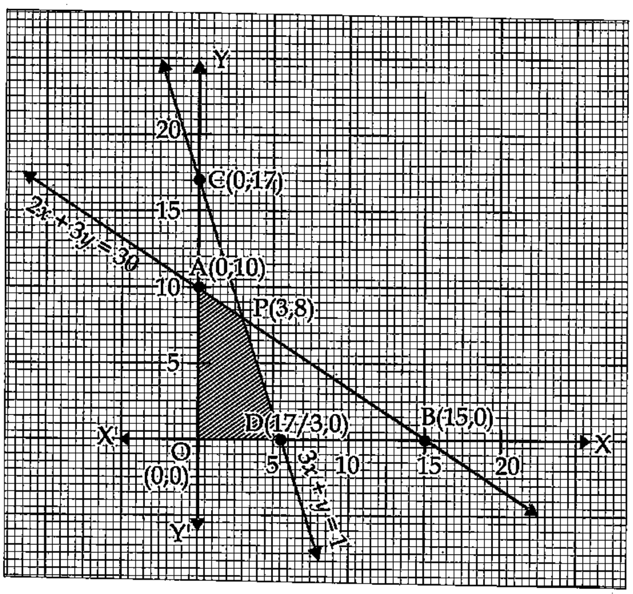
First we draw the line AB and CD whose equations are

$$2x + 3y = 30 \quad \ldots(i)$$

	A	B
x	0	15
y	10	0

and $$3x + y = 17 \quad \ldots(ii)$$

	C	D
x	0	17/3
y	17	0



The feasible region is ODPAO which is shaded in the figure.

P is the point of intersection of the lines

$$2x + 3y = 30$$

and $\quad 3x + y = 17$

Solving these equations, we get point P(3, 8).

The vertices of the feasible region are O(0, 0), D(17/3), P (3, 8) and A(0, 10).

The value of objective function $Z = 100x + 120y$ at these vertices are as follows :

Corner Point	Total Revenue $Z = 100x + 120y$
At O (0, 0)	$Z = 0$
At D (17/3, 0)	$Z = \frac{1700}{3}$
At P(3, 8)	$Z = 1260 \leftarrow$ maximum
At A (0, 10)	$Z = 1200$

$\therefore$ The maximum revenue ₹ 1260 at the point P(3, 8) *i.e.*, when 3 units of goods A and 8 units of goods B are produced.

Yes, I agree with the view of the manufacturer.

Ans.

29. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some others (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others using matrix method, find the number of awardees of each category. Apart from these values namely honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards. [6]

Solution : From question,

$$x + y + z = 12 \quad \text{...(i)}$$

$$2x + 3(y + z) = 33 \Rightarrow 2x + 3y + 3z = 33 \quad \text{...(ii)}$$

$$x + z = 2y \Rightarrow x - 2y + z = 0 \quad \text{...(iii)}$$

$\therefore$ The given equations can be written in matrix form

$$AX = B \quad \text{...(iv)}$$

Here $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 1(3 + 6) - 1(2 - 3) + 1(-4 - 3)$$

$$= 9 + 1 - 7 = 3 \neq 0$$

$\Rightarrow A^{-1}$ exists.

For adj A,

$A_{11} = (3 + 6) = 9, A_{12} = -(2 - 3) = 1, A_{13} = (-4 - 3) = -7$

$A_{21} = -(1 + 2) = -3, A_{22} = (1 - 1) = 0, A_{23} = -(-2 - 1) = 3$

$A_{31} = (3 - 3) = 0, A_{32} = -(3 - 2) = -1, A_{33} = (3 - 2) = 1$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 9 & 1 & -7 \\ -3 & 0 & 3 \\ 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj. } A = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$\therefore$ From (iv), $\quad X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 108 - 99 + 0 \\ 12 + 0 + 0 \\ -84 + 99 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad x = 3, \; y = 4, z = 5$$

The colony management must includes cleanliness for awards. **Ans.**

Mathematics 2013 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set

SECTION — A

9. If matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = pA$, then write the value of p. **[1]**

Solution : Given that

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 4+4 & -4-4 \\ -4-4 & 4+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = 4\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 4A$$

Therefore, the value of p is 4. **Ans.**

10. A and B are two points with position vectors $2\vec{a} - 3\vec{b}$ and $6\vec{b} - \vec{a}$ respectively. Write the position vector of a point P which divides the line segment AB internally in the ratio 1 : 2. [1]

Solution : Position vector of point

$$P = \frac{2(2\vec{a} - 3\vec{b}) + 1(6\vec{b} - \vec{a})}{1+2}$$

$$= \frac{4\vec{a} - 6\vec{b} + 6\vec{b} - \vec{a}}{3}$$

$$= \frac{3\vec{a}}{3} = \vec{a}$$ **Ans.**

SECTION — B

19. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$. **[4]**

Solution : Given, $x^y = e^{x-y}$

Taking log on both sides, we get

$y \log x = x - y \Rightarrow x = y(1 + \log x)$

$$\Rightarrow \quad y = \frac{x}{1+\log x}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x).1 - x.\frac{1}{x}}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}.$$ **Hence Proved**

20. Evaluate : $\int \frac{dx}{x(x^3+8)}$. **[4]**

Solution :

$$\int \frac{dx}{x(x^3+8)} = \int \frac{\frac{1}{x^3}}{\frac{x(x^3+8)}{x^3}} dx = \int \frac{dx}{x^4\left(1+\frac{8}{x^3}\right)}$$

Let, $\left(1+\frac{8}{x^3}\right) = t \Rightarrow -\frac{24}{x^4}dx = dt$

$$\Rightarrow \quad \frac{dx}{x^4} = -\frac{dt}{24}$$

$$\therefore \quad I = -\frac{1}{24}\int \frac{dt}{t} = -\frac{1}{24}\log|t|$$

$$= -\frac{1}{24}\log\left|\left(1+\frac{8}{x^3}\right)\right| + C$$

$$= \frac{1}{24}\log\left|\frac{x^3}{x^3+8}\right| + C.$$ **Ans.**

21. Evaluate : $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$. **[4]**

Solution :

Let $\quad I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ ...(i)

and $\quad I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$

$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx$$

Let, $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\Rightarrow \quad \sin x\, dx = -dt$

when $x = 0, t = 1$ and $x = \pi, t = -1$

$$\Rightarrow \quad 2I = -\int_1^{-1} \frac{\pi}{1+t^2} dt = -\pi\left[\tan^{-1} t\right]_1^{-1}$$

$$= -\pi\,[\tan^{-1}(-1) - \tan^{-1}(1)]$$

$$= -\pi\left(-\frac{\pi}{4}-\frac{\pi}{4}\right) = -\pi\left(-\frac{\pi}{2}\right)$$

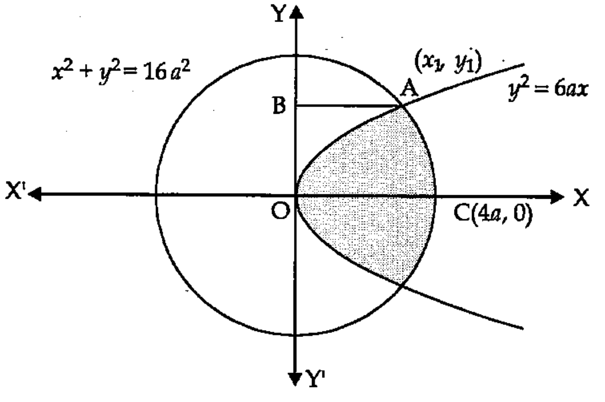
$$= \frac{\pi^2}{2}$$

$$\therefore \quad I = \frac{\pi^2}{4}.$$ **Ans.**

SECTION — C

28. Find the area of the region $\{(x, y) : y^2 \le 6ax$ and $x^2 + y^2 \le 16a^2\}$ using method of integration. [6]

Solution :



Coordinates of point C is $(4a, 0)$. Let the coordinates of point A be (x_1, y_1).

$\therefore x_1^2 + 6ax_1 = 16a^2 \Rightarrow (x_1 - 2a)(x_1 + 8a) = 0$

$\Rightarrow x_1 = 2a, x_1 \ne -8a$ as x_1 lies in the first quadrant.

$\therefore y_1^2 = 6ax_1 = 12a^2 \quad \Rightarrow y_1 = 2\sqrt{3}.a$

Required Area = Area of the shaded region

$$= 2\left(\left|\int_0^{2\sqrt{3}.a}\left(\sqrt{16a^2 - y^2}\right)dy\right| - \left|\int_0^{2\sqrt{3}.a}\left(\frac{y^2}{6a}\right)dy\right|\right)$$

$$= 2\left[\left(2\sqrt{3}a^2 + \frac{8\pi a^2}{3}\right) - \left(\frac{4\sqrt{3}}{3}a^2\right)\right]$$

$$= \frac{4a^2}{3}(4\pi + \sqrt{3}) \text{ sq. units.}$$ **Ans.**

29. Show that the differential equation $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$ is homogeneous.

Find the particular solution of this differential equation, given that $y = \frac{\pi}{4}$ when $x = 1$. [6]

Solution : Given, $\left[x\sin^2\left(\frac{y}{x}\right) - y\right]dx + xdy = 0$

$$\Rightarrow \quad x\frac{dy}{dx} = y - x\sin^2\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x}$$

which is a homogeneous differential equation.

Put, $\quad y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$,

we get

$$v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

Separating variables and integrating

$$\int \frac{dv}{\sin^2 v} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad -\cot v = -\log x + C$$

$$\Rightarrow \quad \log x - \cot(y/x) = C$$

When $x = 1, y = \frac{\pi}{4}$

$$\log 1 - \cot\frac{\pi}{4} = C$$

$$\Rightarrow \quad 0 - 1 = C \Rightarrow C = -1$$

Hence required particular solution is

$$\log x - \cot\left(\frac{y}{x}\right) + 1 = 0$$ **Ans.**

Mathematics 2013 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — A

9. If matrix $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = \lambda A$, then write the value of λ. [1]

Solution : Given,

$$A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \text{and } A^2 = \lambda A$$

$$\Rightarrow \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \lambda\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} = \lambda \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\Rightarrow 6\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \lambda \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\Rightarrow 6A = \lambda A$$

$$\Rightarrow \lambda = 6.$$ **Ans.**

10. L and M are two points with position vectors $2\vec{a}-\vec{b}$ and $\vec{a}+2\vec{b}$ respectively. Write the position vector of a point N which divides the line segment LM in the ratio 2 : 1 externally. **[1]**

Solution : The position vectors of the points are L $(2\vec{a}-\vec{b})$ and M $(\vec{a}+2\vec{b})$.

We have to divide segment LM at N externally in the ratio 2 : 1 .

Position of vector of point N

$$= \frac{2(\vec{a}+2\vec{b})-1(2\vec{a}-\vec{b})}{2-1}$$

$$= 2\vec{a}+4\vec{b}-2\vec{a}+\vec{b}$$

$$= 5\vec{b}$$ **Ans.**

SECTION — B

19. Using vectors, find the area of the triangle ABC, whose vertices are A(1, 2, 3), B(2, –1, 4) and C(4, 5, –1). **[4]**

Solution : Given,

$$A(1,2,3) \Rightarrow \vec{OA} = \hat{i}+2\hat{j}+3\hat{k}$$

$$B(2,-1,4) \Rightarrow \vec{OB} = 2\hat{i}-\hat{j}+4\hat{k}$$

$$C(4,5,-1) \Rightarrow \vec{OC} = 4\hat{i}+5\hat{j}-\hat{k}$$

$$\therefore \vec{AB} = \vec{OB} - \vec{OA}$$

$$= (2\hat{i}-\hat{j}+4\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$$

$$= \hat{i}-3\hat{j}+\hat{k}$$

$$\vec{AC} = \vec{OC}-\vec{OA}$$

$$= (4\hat{i}+5\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$$

$$= 3\hat{i}+3\hat{j}-4\hat{k}$$

$$\text{Now } (\vec{AB}\times\vec{AC}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$$

$$= \hat{i}(12-3)-\hat{j}(-4-3)+\hat{k}(3+9)$$

$$= 9\hat{i}+7\hat{j}+12\hat{k}$$

$$\therefore |\vec{AB}\times\vec{AC}| = \sqrt{(9)^2+(7)^2+(12)^2}$$

$$= \sqrt{81+49+144}$$

$$= \sqrt{274}$$

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AC}|$$

$$= \frac{1}{2}\sqrt{274} \text{ sq. units}$$ **Ans.**

20. Evaluate : $\int \frac{dx}{x(x^3+1)}$. **[4]**

Solution :

$$\text{Let } I = \int \frac{1}{x(x^3+1)}dx$$

$$= \int \frac{x^2}{x^3(x^3+1)}dx$$

$$\text{Put } x^3+1 = t$$

$$\Rightarrow x^3 = t-1$$

$$\Rightarrow 3x^2\,dx = dt$$

$$\Rightarrow x^2dx = \frac{dt}{3}$$

$$\therefore I = \frac{1}{3}\int \frac{1}{t(t-1)}dt$$

$$= \frac{1}{3}\int \frac{1}{t^2-t}dt$$

$$= \frac{1}{3}\int \frac{1}{t^2-t+\frac{1}{4}-\frac{1}{4}}dt$$

$$= \frac{1}{3}\int \frac{1}{\left(t-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{3}\times\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{t-\frac{1}{2}-\frac{1}{2}}{t-\frac{1}{2}+\frac{1}{2}}\right|+C$$

$$= \frac{1}{3}\log\left|\frac{t-1}{t}\right|+C$$

$$\therefore I = \frac{1}{3}\log\left|\frac{x^3}{x^3+1}\right|+C.$$ **Ans.**

21. If $x\sin(a+y)+\sin a\cos(a+y)=0$, prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$. **[4]**

Solution : Here, $x \sin(a+y) + \sin a \cos(a+y) = 0$

$$\Rightarrow \quad x = -\sin a . \frac{\cos(a+y)}{\sin(a+y)}$$

$$\Rightarrow \quad x = -\sin a . \cot(a+y)$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = -\sin a . \{-\operatorname{cosec}^2(a+y).(0+1)\}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{\sin^2(a+y)}{\sin a}.$$

Hence Proved.

22. Using properties of determinants prove the following : **[4]**

$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3(x+y+z)(xy+yz+xz)$$

Solution :

Let
$$\Delta = \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & -y+z \\ x+y+z & y-z & 3z \end{vmatrix}$$

Taking $(x+y+z)$ common from C_1, we get

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & -y+z \\ 1 & y-z & 3z \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\therefore \quad = (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & -y+x \\ 0 & -z+x & 2z+x \end{vmatrix}$$

Expanding along C_1, we get

$$= (x+y+z)\begin{vmatrix} 2y+x & -y+x \\ -z+x & 2z+x \end{vmatrix}$$

$$= (x+y+z)[(2y+x)(2z+x) - (-y+x)(-z+x)]$$

$$= (x+y+z)[(4yz+2xy+2xz+x^2) - (yz-xy-xz+x^2)]$$

$$= (x+y+z)(3yz+3xy+3xz)$$

$$\therefore \quad = 3(x+y+z)(xy+yz+zx).$$

Hence Proved.

28. Find the area of the region, $\{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$ using method of integration. **[6]**

Solution : Given

$$R = \{(x, y) : y^2 \le 4x, 4x^2 + 4y^2 \le 9\}$$

$$\Rightarrow \quad R = \{(x, y) : y^2 \le 4x\} \cap \{(x, y) : 4x^2 + 4y^2 \le 9\}$$

$$\Rightarrow \quad R = R_1 \cap R_2,$$

Where $R_1 = \{(x, y) : y^2 \le 4x\}$ ...(i)

$R_2 = \{(x, y) : 4x^2 + 4y^2 \le 9\}$ ...(ii)

$$y^2 = 4x$$

$$4x^2 + 4y^2 = 9$$

$$\Rightarrow \quad 4x^2 + 16x - 9 = 0$$

$$\Rightarrow \quad (2x+9)(2x-1) = 0$$

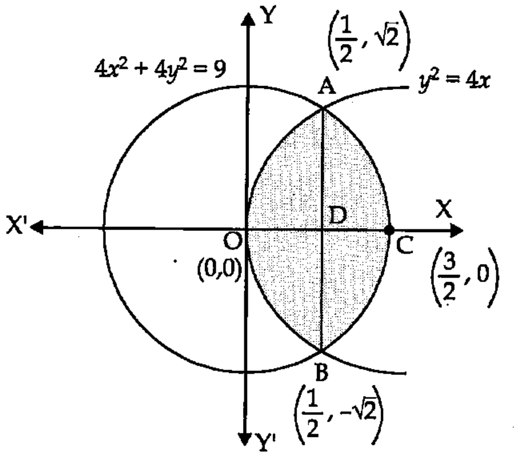
$$\Rightarrow \quad x = \frac{-9}{2} \text{ or } x = \frac{1}{2}$$

From equation (i), we find that $x = \frac{1}{2}$

$$\Rightarrow \quad y = \pm\sqrt{2} \text{ and } x = \frac{-9}{2}$$

$\Rightarrow y$ is imaginary.

So the two curves intersect at $\left(\frac{1}{2}, \sqrt{2}\right)$ and $\left(\frac{1}{2}, -\sqrt{2}\right)$



Required area = Area of shaded region

= 2 (Area in Ist Quadrant)

= 2 (Area of OADO + Area of ADCA)

$$A = 2\int_0^{\frac{1}{2}} 2\sqrt{x}\,dx + 2\int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4} - x^2}\,dx$$

$$= 4 \times \frac{2}{3}\left[x^{3/2}\right]_0^{1/2} + 2\left[\frac{1}{2}x\sqrt{\frac{9}{4} - x^2} + \frac{1}{2}.\frac{9}{4}\sin^{-1}\frac{2x}{3}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{8}{3}\left(\frac{1}{2\sqrt{2}} - 0\right) + \left[\left\{\frac{9}{4}\sin^{-1}(1)\right\} - \left\{\frac{1}{\sqrt{2}} + \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= \frac{2\sqrt{2}}{3} + \left[\frac{9}{8}\pi - \frac{1}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$\therefore \quad A = \left[\frac{\sqrt{2}}{6}+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right]$ sq. units.

Ans.

29. Find the particular solution of the differential equation $\frac{dx}{dy}+x\cot y=2y+y^2\cot y, (y\neq 0)$, given that $x=0$ when $y=\frac{\pi}{2}$. **[6]**

Solution : The given differential equation is

$$\frac{dx}{dy}+x\cot y=2y+y^2\cot y, (y\neq 0)$$

Given equation is of the form $\frac{dx}{dy}+Px=Q$

Here, $P=\cot y$

and $Q=2y+y^2\cot y$

$\Rightarrow \quad I.F. = e^{\int P dy}=e^{\int \cot y\, dy}$

$= e^{\log|\sin y|}=\sin y$

Now, solution is given by

$x\,(I.F.) = \int (I.F.)\, Q\, dy+C$

$\Rightarrow \quad x\sin y = \int \sin y(2y+y^2\cot y)dy+C$

$= \int(2y\sin y+y^2\cot y\sin y)dy+C$

$= \int 2y\sin y\, dy+\int y^2\cos y\, dy+C$

$= 2\sin y.\frac{y^2}{2}-2\int\cos y.\frac{y^2}{2}\, dy+\int y^2\cos y\, dy+C$

$= y^2\sin y-\int y^2\cos y\, dy+\int y^2\cos y dy+C$

$= y^2\sin y+C$

$\therefore x\sin y = y^2\sin y+C$ ...(i)

Given that $x=0$, when $y=\frac{\pi}{2}$.

$\Rightarrow (0).\sin\left(\frac{\pi}{2}\right)=\left(\frac{\pi}{2}\right)^2\sin\left(\frac{\pi}{2}\right)+C$

$\Rightarrow \quad 0 = \frac{\pi^2}{4}+C$

$\therefore \quad C = \frac{-\pi^2}{4}$

Hence the required particular solution of given differential equation is

$x\sin y = y^2\sin y-\frac{\pi^2}{4}$. **Ans.**

Mathematics 2013 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. Write the principal value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$. **[1]**

Solution : $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)$

$=\frac{\pi}{4}+\left[\pi-\cos^{-1}\left(\frac{1}{2}\right)\right]$

$[\because \cos^{-1}(-x)=\pi-\cos^{-1}x]$

$=\frac{\pi}{4}+\pi-\frac{\pi}{3}=\frac{\pi}{4}+\frac{2\pi}{3}=\frac{11}{12}\pi$. **Ans.**

2. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$. **[1]**

Solution : $\tan\left(2\tan^{-1}\frac{1}{5}\right)=\tan\left(\tan^{-1}\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right)$

$\left(\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right)$

$=\tan\left(\tan^{-1}\frac{\frac{2}{5}}{\frac{24}{25}}\right)$

$=\tan\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$

$=\frac{5}{12}$. **Ans.**

3. Find the value of a if $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\ 0 & 13\end{bmatrix}$. **[1]**

Solution : Given, $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\ 0 & 13\end{bmatrix}$

Comparing the corresponding elements, we get

$a-b=-1$ ...(i)

$2a-b=0$ ...(ii)

$2a+c=5$ ...(iii)

$3c+d=13$ ...(iv)

Solving equation (i) and (ii), we get

$a=1$. **Ans.**

4. **If** $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$**, then write the value of** x. **[1]**

Solution : Given, $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$

$\Rightarrow (x+1)(x+2) - (x-3)(x-1)$
$= 4 \times 3 - 1 \times (-1)$

$\Rightarrow (x^2 + 3x + 2) - (x^2 - 4x + 3)$
$= 12 + 1$

$\Rightarrow \quad 3x + 2 + 4x - 3 = 13$

$\Rightarrow \quad 7x - 14 = 0$

$\Rightarrow \quad x = 2.$ **Ans.**

5. **If** $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$**, then find the matrix A.** **[1]**

Solution : Given $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$$\Rightarrow A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & -1-2 & 4+1 \\ -2-0 & 1-4 & 3-9 \end{bmatrix} = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}.$$

Ans.

6. **Write the degree of the differential equation** $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0.$ **[1]**

Solution : The degree of the given differential equation is 2. **Ans.**

7. **If** $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ **and** $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ **are two equal vectors, then write the value of** $x + y + z$. **[1]**

Solution : Given ; $\vec{a} = \vec{b}$

$\Rightarrow \quad x\hat{i} + 2\hat{j} - z\hat{k} = 3\hat{i} - y\hat{j} + \hat{k}$

Comparing the corresponding element, we get

$x = 3, y = -2, z = -1$

$\therefore \quad x + y + z = 3 + (-2) + (-1) = 0.$

Ans.

8. **If a unit vector** $\vec{a}$ **makes angle** $\frac{\pi}{3}$ **with** $\hat{i}$, $\frac{\pi}{4}$ **with** $\hat{j}$ **and an acute angle θ with** $\hat{k}$ **then find the value of θ.** **[1]**

Solution : $l = \cos\alpha = \cos\frac{\pi}{3} = \frac{1}{2}$

$m = \cos\beta = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$n = \cos\gamma = \cos\theta$

Since, $l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$

$\Rightarrow \quad \cos^2\theta = 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = \frac{\pi}{3}.$ **Ans.**

9. **Find the cartesian equation of the line which passes through the point (– 2, 4, – 5) and is parallel to the line** $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$. **[1]**

Solution : Direction ratios of the line parallel to

$\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$ *i.e.,* $\frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+8}{6}$ ...(i)

are 3, – 5, 6.

$\therefore$ Cartesian equation of the lines passes through (– 2, 4, – 5) and parallel to line (i) is

$\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}.$ **Ans.**

10. **The amount of pollution content added in air in a city to** x**-diesel vehicles is given by** $P(x) = 0.005x^3 + 0.02x^2 + 30x$**. Find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above question.** **[1]**

Solution : Here, pollution content is given by

$P(x) = 0.005x^3 + 0.02x^2 + 30x$

where x is the number of diesel vehicles.

$\Rightarrow \quad \frac{dP}{dx} = 0.015x^2 + 0.04x + 30$

$\therefore$ The marginal increase in pollution content (when $x = 3$)

$= 0.015 \times (3)^2 + 0.04 \times 3 + 30$

$= 0.135 + 0.12 + 30$

$= 30.255$

The value indicated in the question is diesel vehicles causes environmental pollution. **Ans.**

SECTION — B

11. **Show that the function** f **in** $A = R - \left\{\frac{2}{3}\right\}$ **defined as** $f(x) = \frac{4x+3}{6x-4}$ **is one-one and onto. Hence find** f^{-1}. **[4]**

Solution : Given, $f(x) = \frac{4x+3}{6x-4}$ where $x \in A$

$= R - \left\{\frac{2}{3}\right\}$

$f(x_1) = f(x_2)$ where $(x_1, x_2 \in A)$

$\Rightarrow \quad \dfrac{4x_1+3}{6x_1-4} = \dfrac{4x_2+3}{6x_2-4}$

$\Rightarrow \quad (4x_1+3)(6x_2-4) = (6x_1-4)(4x_2+3)$

$\Rightarrow 24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$

$\Rightarrow \quad -34x_1 = -34x_2 \Rightarrow x_1 = x_2.$

$\therefore f$ is one-one. **Hence Proved.**

For $\quad y \in A = R - \left\{\dfrac{2}{3}\right\}$

$f(x) = y$

$\Rightarrow \quad \dfrac{4x+3}{6x-4} = y \Rightarrow (6x-4)y = 4x+3$

$\Rightarrow \quad 6xy - 4y = 4x + 3$

$\Rightarrow \quad (6y-4)x = 4y+3$

$\Rightarrow \quad x = \dfrac{4y+3}{6y-4}$

$\Rightarrow f$ is onto. **Hence Proved.**

Hence f is invertible as it is one-one and onto.

Now, $\quad f(x) = y \Leftrightarrow x = f^{-1}(y)$

$\Rightarrow \quad f^{-1}(y) = \dfrac{4y+3}{6y-4} \forall y \in A.$ **Ans.**

12. Find the value of the following :

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], \ |x| < 1, y > 0$$

and $xy < 1$. **[4]**

Solution : $\tan\dfrac{1}{2}\left[\sin^{-1}\dfrac{2x}{1+x^2} + \cos^{-1}\dfrac{1-y^2}{1+y^2}\right],$

Putting $x = \tan\alpha$ and $y = \tan\beta$

$$\tan\frac{1}{2}\left[\sin^{-1}\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right) + \cos^{-1}\left(\frac{1-\tan^2\beta}{1+\tan^2\beta}\right)\right]$$

$$= \tan\frac{1}{2}[\sin^{-1}(\sin 2\alpha) + \cos^{-1}(\cos 2\beta)]$$

$$\left[\because \sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha} \text{ and } \cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right]$$

$$= \tan\frac{1}{2}(2\alpha + 2\beta) = \tan(\alpha+\beta)$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{x+y}{1-xy}.$$ **Ans.**

OR

Prove that :

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}.$$

Solution : L.H.S.

$$= \left[\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right)\right] + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\cdot\frac{1}{5}}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$\left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \tan^{-1}\left(\frac{7}{10}\cdot\frac{10}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{7}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\cdot\frac{1}{8}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{65}{72}}{\frac{65}{72}}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S.}$$ **Hence Proved.**

13. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2.$$

[4]

Solution : L.H.S. $= \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix} 1-x & x-x^2 & x^2 \\ x^2-1 & 1-x & x \\ x-x^2 & x^2-1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-x & x(1-x) & x^2 \\ (x-1)(x+1) & 1-x & x \\ x(1-x) & (x-1)(x+1) & 1 \end{vmatrix}$$

Taking $(1-x)$ common from C_1 and C_2, we get.

$$= (1-x).(1-x)\begin{vmatrix} 1 & x & x^2 \\ -(x+1) & 1 & x \\ x & -(x+1) & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$= (1-x)^2\begin{vmatrix} 0 & 0 & x^2+x+1 \\ -(x+1) & 1 & x \\ x & -(x+1) & 1 \end{vmatrix}$$

Expanding along R_1, we get

$$= (1-x)^2.(1+x+x^2)\,[(x+1)^2 - x]$$
$$= (1-x)^2\,(1+x+x^2)\,(1+x+x^2)$$
$$= [(1-x)\,(1+x+x^2)]^2$$
$$= (1-x^3)^2 = \text{R.H.S.}$$
$$[\because a^3 - b^3 = (a-b)\;(a^2+ab+b^2)]$$

Hence Proved.

14. Differentiate the following function with respect to x:

$$(\log x)^x + x^{\log x} \qquad [4]$$

Solution : Let, $y = (\log x)^x + x^{\log x}$

$= u + v$, where $u = (\log x)^x$ and $v = x^{\log x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{...(i)}$$

Now, $u = (\log x)^x$

Taking log on both sides, we get

$$\log u = x.\log(\log x)$$

Differentiating w.r. t. x, we get

$$\frac{1}{u}\frac{du}{dx} = 1.\log(\log x) + x.\frac{1}{\log x}.\frac{1}{x}$$

$$\Rightarrow \quad \frac{du}{dx} = u\left[\log(\log x) + \frac{1}{\log x}\right] \qquad \text{...(ii)}$$

and $v = x^{\log x}$

Taking log on both sides, we get

$$\log v = \log x.\log x = (\log x)^2$$

Differentiating w.r. t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = 2.\log x.\frac{1}{x}$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left[2\log x.\frac{1}{x}\right]$$

$$\Rightarrow \quad \frac{dv}{dx} = x^{\log x}.\frac{2\log x}{x} \qquad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right] + x^{\log x}.\frac{2\log x}{x} \quad \textbf{Ans.}$$

15. If $y = \log\left[x + \sqrt{x^2+a^2}\right]$, show that

$$(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0. \qquad [4]$$

Solution : Given, $y = \log\left[x + \sqrt{x^2+a^2}\right]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}}.\frac{d}{dx}\left[x+\sqrt{x^2+a^2}\right]$$
$$= \frac{1}{x+\sqrt{x^2+a^2}}\left[1 + \frac{1}{2}(x^2+a^2)^{-1/2}.2x\right]$$
$$= \frac{1}{x+\sqrt{x^2+a^2}}\left[1 + \frac{x}{\sqrt{x^2+a^2}}\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}}.\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}} = \frac{1}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \sqrt{x^2+a^2}.\frac{dy}{dx} = 1$$

Again differentiating w.r. t. x, we get

$$\sqrt{x^2+a^2}.\frac{d^2y}{dx^2} + \frac{x}{\sqrt{x^2+a^2}}.\frac{dy}{dx} = 0$$

Multiply by $\sqrt{x^2+a^2}$, we get

$$(x^2+a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

Hence Proved.

16. Show that the function $f(x) = |x-3|, x \in$ R, is continuous but not differentiable at $x = 3$. [4]

Solution : Given, $f(x) = |x-3|, x \in \text{R}$

$$= \begin{cases} x-3, & \text{if } x \geq 3 \\ 3-x, & \text{if } x < 3 \end{cases}$$

When $x > 3, f(x) = x - 3$ and it is a polynomial, so it is continuous.

When $x < 3, f(x) = 3 - x$. Again it is a polynomial, so it is continuous.

Also $f(3-0) = 0 = f(3+0) = f(3)$

$\Rightarrow f(x)$ is continuous at $x = 3$.

$$\text{Now, LHD} = f'(3-0) = \lim_{x\to 3-h}\frac{f(x)-f(3)}{-h}$$
$$= \lim_{h\to 0}\frac{3-(3-h)-0}{-h}$$
$$= \lim_{h\to 0}\frac{h}{-h} = -1.$$

$$\text{and} \quad \text{RHD} = f'(3+0) = \lim_{x\to 3+h}\frac{f(x)-f(3)}{h}$$
$$= \lim_{h\to 0}\frac{(3+h)-3-0}{h}$$
$$= \lim_{h\to 0}\frac{h}{h} = 1.$$

Since LHD $\neq$ RHD

$\Rightarrow f$ is not differentiable at $x = 3$. **Hence Proved.**

OR

If $x = a \sin t$ and $y = a(\cos t + \log \tan t/2)$, find $\frac{d^2y}{dx^2}$.

Solution : Given, $x = a \sin t$

Differentiating w.r. t. x, we get

$$\frac{dx}{dt} = a \cos t \quad ...(i)$$

and $$y = a\left(\cos t + \log \tan \frac{t}{2}\right)$$

Differentiating w.r. t. t, we get

$$\frac{dy}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}} \sec^2 \frac{t}{2}.\frac{1}{2}\right)$$

$$\Rightarrow \frac{dy}{dt} = a\left[-\sin t + \frac{\cos t/2}{\sin t/2} \times \frac{1}{2\cos^2 t/2}\right]$$

$$\Rightarrow \frac{dy}{dt} = a\left[-\sin t + \frac{1}{2\sin t/2 \cos t/2}\right]$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$[\because \sin 2\theta = 2 \sin\theta \cos\theta]$

$$= a.\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$= a.\frac{\cos^2 t}{\sin t} \quad ...(ii)$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= a.\frac{\cos^2 t}{\sin t} / a\cos t$$

[Using (i) and (ii)]

$$\therefore \frac{dy}{dx} = \cot t.$$

Differentiating w.r. t. x, we get

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2 t.\frac{dt}{dx}$$

$$= -\operatorname{cosec}^2 t / \frac{dx}{dt}$$

$$= -\operatorname{cosec}^2 t / a\cos t \quad \text{[Using (i)]}$$

$$= -\frac{1}{a}\sec t \operatorname{cosec}^2 t.$$ **Ans.**

17. Evaluate : $\int \frac{\sin(x-a)}{\sin(x+a)}dx.$ **[4]**

Solution : Let, $I = \int \frac{\sin(x-a)}{\sin(x+a)}dx$

$$= \int \frac{\sin(x+a-a-a)}{\sin(x+a)}dx$$

$$= \int \frac{\sin[(x+a)-2a]}{\sin(x+a)}dx$$

$$= \int \frac{\sin(x+a)\cos 2a - \cos(x+a)\sin 2a}{\sin(x+a)}dx$$

[Using formula of sin (A – B)]

$$= \int [\cos 2a - \sin 2a.\cot(x+a)]dx$$

$$= \cos 2a \int 1.dx - \sin 2a \int \cot(x+a)dx$$

$$= x\cos 2a - \sin 2a \log|\sin(x+a)| + C.$$ **Ans.**

OR

Evaluate : $\int \frac{5x-2}{1+2x+3x^2}dx.$

Solution : Let $I = \int \frac{5x-2}{1+2x+3x^2}dx$

Now, $$5x-2 = A\frac{d}{dx}(1+2x+3x^2)+B$$

$$\Rightarrow 5x-2 = A(2+6x)+B$$

$$\Rightarrow 5x-2 = 6Ax+2A+B$$

On equating the coefficient of x and constant on both sides, we get

$$5 = 6A \Rightarrow A = 5/6$$

and $$2A+B = -2 \Rightarrow 2\times\frac{5}{6}+B = -2$$

$$\Rightarrow B = -2-\frac{5}{3} = -\frac{11}{3}$$

$$\therefore 5x-2 = \frac{5}{6}(2+6x)+\left(-\frac{11}{3}\right)$$

$$\therefore I = \int \frac{5x-2}{1+2x+3x^2}dx$$

$$= \int \frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}dx$$

$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$$

$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3.3}\int \frac{dx}{x^2+\frac{2}{3}x+\frac{1}{3}}$$

$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}$$

$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{9}\cdot\frac{1}{\sqrt{2}/3}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\sqrt{2}/3}\right) + C$$

$$= \frac{5}{6}\log|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$ **Ans.**

18. Evaluate : $\int \frac{x^2}{(x^2+4)(x^2+9)}dx.$ **[4]**

Solution : Here, $\frac{x^2}{(x^2+4)(x^2+9)} = \frac{y}{(y+4)(y+9)}$,

where $y = x^2$

Let $\frac{y}{(y+9)(y+4)} = \frac{A}{y+4} + \frac{B}{y+9}$

$\Rightarrow y = A(y+9) + B(y+4)$

Putting $y = -9$ and -4, we get

$-4 = A(-4+9)$ and $-9 = B(-9+4)$

$\Rightarrow B = \frac{9}{5}$ and $A = -\frac{4}{5}$

$\therefore \frac{y}{(y+4)(y+9)} = -\frac{4}{5}\cdot\frac{1}{y+4} + \frac{9}{5}\cdot\frac{1}{y+9}$

$\Rightarrow \frac{x^2}{(x^2+4)(x^2+9)} = -\frac{4}{5}\cdot\frac{1}{x^2+4} + \frac{9}{5}\cdot\frac{1}{x^2+9}$

On integrating both sides w.r.t. x, we get

$$\int \frac{x^2}{(x^2+4)(x^2+9)}dx = -\frac{4}{5}\int\frac{dx}{x^2+2^2} + \frac{9}{5}\int\frac{dx}{x^2+3^2}$$

$$= -\frac{4}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{9}{5}\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= -\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right) + C.$$ **Ans.**

19. Evaluate : $\int_0^4 (|x|+|x-2|+|x-4|)dx.$ **[4]**

Solution : Let $I = \int_0^4 (|x|+|x-2|+|x-4|)dx$

$$= \int_0^2 (|x|+|x-2|+|x-4|)dx + \int_2^4 (|x|+|x-2|+|x-4|)dx$$

$$= \int_0^2 [x-(x-2)-(x-4)]dx + \int_2^4 [x+(x-2)-(x-4)]dx$$

$$= \int_0^2 (x-x+2-x+4)dx + \int_2^4 (x+x-2-x+4)dx$$

$$= \int_0^2 (-x+6)dx + \int_2^4 (x+2)dx$$

$$= \left[-\frac{x^2}{2}+6x\right]_0^2 + \left[\frac{x^2}{2}+2x\right]_2^4$$

$= (-2+12-0) + [8+8-(2+4)]$

$= 10 + 10 = 20.$ **Ans.**

20. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}| = |\vec{a}|$, then prove that vector $2\vec{a}+\vec{b}$ is perpendicular to vector $\vec{b}$. **[4]**

Solution : Given,

$$|\vec{a}+\vec{b}| = |\vec{a}|$$

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{a}|^2$

$\Rightarrow \quad (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a}$

$\Rightarrow \quad \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{a}$

$\Rightarrow \quad 2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b} = 0$

$(\because \vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b})$

$\Rightarrow \quad (2\vec{a}+\vec{b})\cdot\vec{b} = 0$

$\Rightarrow \quad (2\vec{a}+\vec{b}) \perp \vec{b}.$

$\therefore$ The vector $2\vec{a}+\vec{b}$ is perpendicular to vector $\vec{b}$.

Hence Proved.

21. Find the coordinates of the point, where the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$ intersects the plane $x - y + z - 5 = 0$. Also find the angle between the line and the plane. **[4]**

Solution : Equation of the line is

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = k \text{ (say)} \quad ...(i)$$

$\therefore$ Point on line is $(3k+2, 4k-1, 2k+2)$.

If it lies on the plane $\quad x - y + z - 5 = 0 \quad ...(ii)$

then $\quad (3k+2) - (4k-1) + (2k+2) - 5 = 0$

$\Rightarrow \quad 3k + 2 - 4k + 1 + 2k - 3 = 0$

$\Rightarrow \quad k = 0$

$\therefore$ Point is $(2, -1, 2)$

$\therefore$ $(2, -1, 2)$ is the point on line (i), where it intersects (ii).

If θ is the angle between line (i) and plane (ii), then

$$\sin\theta = \frac{3.1+4.(-1)+2.1}{\sqrt{3^2+4^2+2^2}.\sqrt{1^2+(-1)^2+1^2}}$$

$$\Rightarrow \quad \sin\theta = \frac{3-4+2}{\sqrt{29}\sqrt{3}} = \frac{1}{\sqrt{87}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$$ **Ans.**

OR

Find the vector equation of the plane which contains the line of intersection of the planes $\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4=0$ and $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8=0$.

Solution : Given planes are

$$\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4 = 0 \quad ...(i)$$

$$\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5 = 0 \quad ...(ii)$$

The vector equation of the plane which contains the line of intersection of the planes (i) and (ii) is

$$\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4+\lambda[\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5] = 0$$

$$\Rightarrow \vec{r}.[(1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}]+(5\lambda-4)=0 \quad ...(iii)$$

Now plane (iii) is perpendicular to plane

$$\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8 = 0 \quad ...(iv)$$

$\therefore$ $(1+2\lambda)\,5+(2+\lambda)\,3+(3-\lambda)\,(-6)=0$

$$\Rightarrow \quad 19\lambda-7=0 \Rightarrow \lambda=\frac{7}{19}.$$

Substituting the value of λ in (iii), we get

$$\vec{r}.\left[\left(1+2\times\frac{7}{19}\right)\hat{i}+\left(2+\frac{7}{19}\right)\hat{j}+\left(3-\frac{7}{19}\right)\hat{k}\right] +\left(5\times\frac{7}{19}-4\right)=0$$

$$\Rightarrow \vec{r}(33\hat{i}+45\hat{j}+50\hat{k})=41.$$ **Ans.**

22. A speaks truth in 60% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact ? In the cases of contradiction do you think, the statement of B will carry more weight as he speaks truth in more number of cases then A ? **[4]**

Solution : Probability of A speaking the truth is

$$P(A) = \frac{60}{100} = \frac{6}{10}$$

$$\Rightarrow \quad (\bar{A}) = 1-P(A) = \frac{4}{10}$$

Probability of B speaking the truth is

$$P(B) = \frac{90}{100} = \frac{9}{10}$$

$$\Rightarrow \quad P(\bar{B}) = 1-P(B) = \frac{1}{10}$$

Now A and B will contradict each other in the following mutually exclusive cases :

(i) A speaks the truth and B does not.
(ii) B speaks the truth and A does not.
By the theorem of total probability
Probability that A and B will contradict each other

$$= P(A).(\bar{B}) + P(B).(\bar{A})$$

$$= \frac{6}{10}.\frac{1}{10}+\frac{9}{10}.\frac{4}{10} = \frac{42}{100}$$

$\therefore$ They will contradict each other in 42% of the cases.

Yes, the statement of B will carry more weight. **Ans.**

SECTION — C

23. A school wants to award its students for the values of Honesty, Regularity and Hard work with a total cash award of ₹ 6,000. Three times the award money for Hard work added to that given for honesty amounts to ₹ 11,000. The award money given for Honesty and Hard work together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from these values namely, Honesty, Regularity and Hard work, suggest one more value which the school must include for awards. **[6]**

Solution : Let award for honesty be ₹ x

award for regularity be ₹ y

and award for hardwork be ₹ z.

According to question,

$$x+y+z = 6000$$

$$x+0y+3z = 11000$$

$$x+z = 2y \Rightarrow x-2y+z=0$$

The given equations can be written in matrix form

$$AX = B \quad ...(i)$$

$$\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 3\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}6000\\ 11000\\ 0\end{bmatrix}$$

Here $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{vmatrix}$

$= 1(0+6) - 1(1-3) + 1(-2-0)$

$= 6 \neq 0$

$\therefore A^{-1}$ exists.

For adj A,

$A_{11} = 0 + 6 = 6,\quad A_{12} = -(1-3) = 2,$

$A_{13} = -2 - 0 = -2,$

$A_{21} = -(1+2) = -3,\quad A_{22} = 1 - 1 = 0,$

$A_{23} = -(-2-1) = 3$

$A_{31} = 3 - 0 = 3,\quad A_{32} = -(3-1) = -2,$

$A_{33} = 0 - 1 = -1$

$$\therefore \text{adj } A = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 3 \\ 3 & -2 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj. } A = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

From (1), $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix}$$

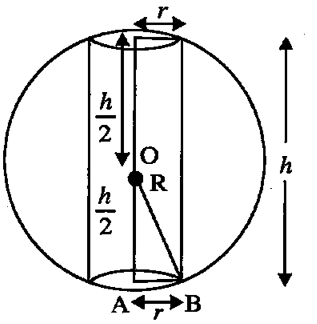
$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

$\Rightarrow x = 500, y = 2000, z = 3500.$

Apart from honesty, regularity and hard work, the school must include an award for a student to be well behaved. **Ans.**

24. Show that the height of the cylinder of maximum volume, that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. [6]

Solution : From the figure:



In Δ OAB,

$$\left(\frac{h}{2}\right)^2 + r^2 = R^2 \quad \text{...(i)}$$

$$\Rightarrow \quad r^2 = R^2 - \frac{h^2}{4}$$

Let, V be Volume of the cylinder inscribed in a sphere

$$\therefore \quad V = \pi r^2 h$$

$$= \pi h\left(R^2 - \frac{h^2}{4}\right) \text{ [using (i)]} \quad \text{...(ii)}$$

$$\therefore \quad V = \pi\left(R^2 h - \frac{h^3}{4}\right)$$

Differentiating w.r.t. h, we get

$$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right) \quad \text{...(iii)}$$

and

$$\frac{d^2V}{dh^2} = \pi\left(0 - \frac{3}{4}.2h\right) \quad \text{...(iv)}$$

For maxima or minima

$$\frac{dV}{dh} = 0$$

From (iii),

$$R^2 - \frac{3}{4}h^2 = 0$$

$$\Rightarrow \quad h^2 = \frac{4}{3}R^2$$

$$\Rightarrow \quad h = \frac{2R}{\sqrt{3}}$$

For the value of h, from (iv),

$$\frac{d^2V}{dh^2} = -\frac{3}{2}\pi.\frac{2R}{\sqrt{3}} = -\sqrt{3}.\pi R < 0(-ve)$$

$\Rightarrow$ V is maximum.

Also maximum value of V

$$= \pi.\frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4}.\frac{4}{3}R^2\right)$$

$$= \pi.\frac{2R}{\sqrt{3}}.\frac{2}{3}R^2 = \frac{4\pi}{3\sqrt{3}}R^3$$

$$= \frac{4\sqrt{3}}{9}\pi R^3 \text{ cu.units} \qquad \textbf{Ans.}$$

OR

Find the equation of the normal at a point on the curve $x^2 = 4y$ which passes through the point (1, 2). Also find the equation of the corresponding tangent.

Solution : The given curve is

$$x^2 = 4y \qquad ...(i)$$

Let (x_1, y_1) be the required point on curve

$$\therefore \quad x_1^2 = 4y_1 \qquad ...(ii)$$

Differentiating eq. (i) w.r.t. x, we get

$$2x = 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

$\therefore$ Slope of tangent $= m = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = \frac{x_1}{2}$

and slope of normal $= m' = -\frac{1}{m} = -\frac{2}{x_1}$

$\therefore$ Eq. of normal at (x_1, y_1) is

$$y - y_1 = m'(x - x_1)$$

It passes through (1, 2)

$$2 - y_1 = -\frac{2}{x_1}(1 - x_1)$$

$$\Rightarrow \quad 2x_1 - x_1 y_1 = -2 + 2x_1$$

$$\Rightarrow \quad x_1 y_1 = 2$$

Put value of y_1 from eq. (ii)

$$x_1 \cdot \frac{x_1^2}{4} = 2$$

$$\Rightarrow \quad x_1^3 = 8 \Rightarrow x_1 = 2$$

$\therefore$ From (ii), $y_1 = \frac{x_1^2}{4} = \frac{2^2}{4} = 1$

$\therefore$ Point on curve is (2, 1) and $m = \frac{2}{2} = 1, m' = -1$

Eq. of normal at (2, 1) and $m' = -1$ is

$$y - 1 = -(x - 2)$$

$$\Rightarrow \quad x + y = 3$$

Eq. of tangent at (2, 1) and $m = 1$

$$y - 1 = (x - 2)$$

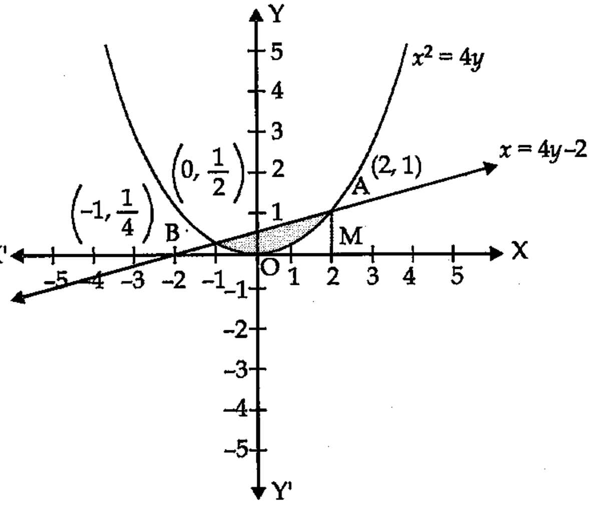
$$\Rightarrow \quad x - y = 1. \qquad \textbf{Ans.}$$

25. Using integration, find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$. **[6]**

Solution : The equation of the given curves are :

$$x^2 = 4y \qquad ...(i)$$

$$x = 4y - 2 \qquad ...(ii)$$



The points of intersection of (i) and (ii) are A (2, 1) and B$\left(-1, \frac{1}{4}\right)$

Required area = Area of shaded region = Area under line – area under parabola.

$$= \int_{-1}^{2} (y_2 - y_1)\,dx$$

$$= \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4}\right) dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right]$$

$$= \frac{1}{4}\left[6 - \frac{8}{3} + \frac{3}{2} - \frac{1}{3}\right] = \frac{1}{4}\left(\frac{9}{2}\right) = \frac{9}{8} \text{ sq. units.} \qquad \textbf{Ans.}$$

OR

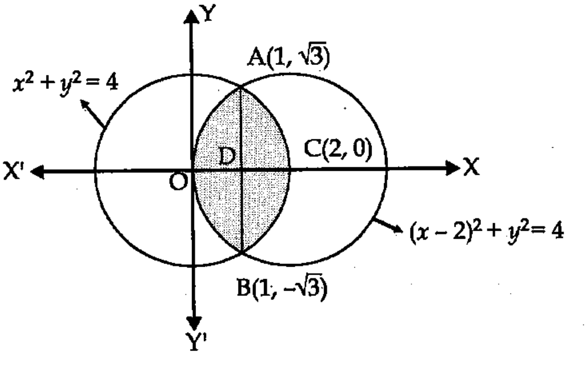
Using integration, find the area of the region enclosed between the two circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

Solution : The given circles are :

$$x^2 + y^2 = 4 \qquad ...(i)$$

and

$$(x - 2)^2 + y^2 = 4 \qquad ...(ii)$$



They meet at $A(1,\sqrt{3})$ and $B(1,-\sqrt{3})$

Required Area = Area of shaded region

= 2 (Area in Ist Quadrant)

= {Area under circle (2) + Area under circle (1)}

$$= 2\left[\int_0^1 y_2 dx + \int_1^2 y_1 dx\right]$$

$$= 2\left[\int_0^1 \sqrt{4-(x-2)^2}\,dx + \int_1^2 \sqrt{4-x^2}\,dx\right]$$

$$= 2\left[\left\{\frac{(x-2)\sqrt{4-(x-2)^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right\}_0^1 + \left\{\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right\}_1^2\right]$$

$$= \left\{-\sqrt{3} + 4\sin^{-1}\left(\frac{-1}{2}\right) - (0 + 4\sin^{-1}(-1))\right\} + \left\{0 + 4\sin^{-1}(1) - \left(\sqrt{3} + 4\sin^{-1}\frac{1}{2}\right)\right\}$$

$$= -\sqrt{3} - 4\sin^{-1}\left(\frac{1}{2}\right) + 4\sin^{-1}(1) + 4\sin^{-1}(1) - \sqrt{3} - 4\sin^{-1}\left(\frac{1}{2}\right)$$

$$= -2\sqrt{3} + 8\sin^{-1}(1) - 8\sin^{-1}\left(\frac{1}{2}\right)$$

$$= 8\left(\frac{\pi}{2} - \frac{\pi}{6}\right) - 2\sqrt{3} = \frac{8}{3}\pi - 2\sqrt{3} \text{ sq. units.}$$ **Ans.**

26. Show that the differential equation $2ye^{x/y}\,dx + (y - 2x e^{x/y})\,dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $x = 0$, when $y = 1$. [6]

Solution : The given differential equation is

$$2ye^{x/y}\,dx + (y - 2x e^{x/y})\,dy = 0$$

Separate the given differential equation, we get

$$\frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}} = \frac{2\frac{x}{y}e^{x/y} - 1}{2e^{x/y}}$$

$$= \lambda^0 f\left(\frac{x}{y}\right)$$

This is a homogeneous differential equation.

Putting $x = vy$

$$\Rightarrow \quad \frac{dx}{dy} = 1.v + y.\frac{dv}{dy}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v}$$

$$= v \frac{1}{2e^v}$$

$$\Rightarrow \quad v + y\frac{dv}{dy} = v - \frac{1}{2e^v}$$

$$\Rightarrow \quad 2e^v\,dv = -\frac{dy}{y}$$

On integrating, we get

$$2\int e^v dv = -\int \frac{dy}{y}$$

$$\Rightarrow \quad 2e^v + \log y = C$$

$$\Rightarrow \quad 2e^{x/y} + \log y = C$$

Put $x = 0, y = 1$, we get

$$2e^0 + \log 1 = C$$

$$\Rightarrow \quad C = 2$$

$\therefore$ The required particular solution is

$$2e^{x/y} + \log y = 2.$$ **Ans.**

27. Find the vector equation of the plane passing through three points with position vectors $\hat{i}+\hat{j}-2\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$. Also find the coordinates of the point of intersection of this plane and the line $\vec{r} = 3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$. [6]

Solution : The given points are

$A(\hat{i}+\hat{j}-2\hat{k}) \equiv (1, 1, -2)$

$B(2\hat{i}-\hat{j}+\hat{k}) \equiv (2, -1, 1)$

and $C(\hat{i}+2\hat{j}+\hat{k}) \equiv (1, 2, 1)$

Equation of any plane passing through A is

$$a(x-1) + b(y-1) + c(z+2) = 0 \quad ...(i)$$

As it is to pass through B and C respectively

$$a - 2b + 3c = 0$$

$$0.a + b + 3c = 0$$

Solving these equations, we get

$$\frac{a}{-6-3} = \frac{b}{0-3} = \frac{c}{1-0}$$

$$\Rightarrow \quad \frac{a}{9} = \frac{b}{3} = \frac{c}{-1}$$

$\therefore$ From (i),

$$9(x-1)+3(y-1)-(z+2)=0$$

$$\Rightarrow \quad 9x+3y-z-14=0 \quad \text{....(ii)}$$

$\therefore$ Vector equation of the plane is

$$\vec{r}.(9\hat{i}+3\hat{j}-\hat{k}) = 14$$

The given line is

$$\vec{r} = 3i-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$$

$$\frac{x-3}{2} = \frac{y+1}{-2} = \frac{z+1}{1} = \lambda \text{ (say)}$$

$(2\lambda+3, -2\lambda-1, \lambda-1)$

If it lies on the above plane (ii), then

$$9(2\lambda+3)+3(-2\lambda-1)-(\lambda-1)-14=0$$

$$\Rightarrow \quad 11\lambda+11=0 \Rightarrow \lambda=-1$$

$\therefore$ Their point of intersection is

$(-2+3, 2-1, -1-1) = (1, 1, -2)$. **Ans.**

28. A cooperative society of farmers has 50 hectares of land to grow two crops A and B. The profits from crops A and B per hectare are estimated as ₹ 10,500 and ₹ 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops A and B at the rate of 20 litre and 10 litre per hectare, respectively. Further not more than 800 litres of herbicide should be used in order to protect fish and wildlife using a pond which collects drainage from this land. Keeping in mind that the protection of fish and other wildlife is more important than earning profit, how much land should be allocated to each crop so as to maximize the total profit ? Form an LPP from the above and solve it graphically. Do you agree with the message that the protection of wildlife is utmost necessary to preserve the balance in environment ? [6]

Solution : Let x hectare and y hectare be allotted to grow crops A and B respectively. Then the L.P.P. is maximize :

$Z = 10{,}500\, x + 9{,}000\, y$ subject to constraints,

$$x+y \le 50$$

$$20x+10y \le 800$$

$$\Rightarrow \quad 2x+y \le 80$$

and $\quad x \ge 0, y \ge 0$

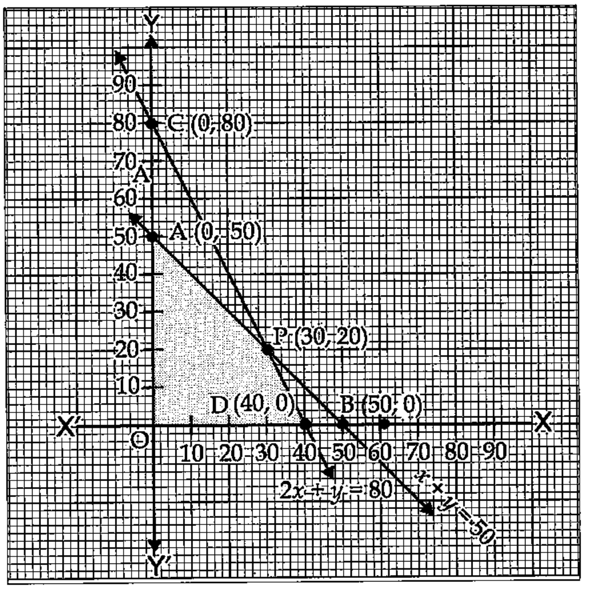
First we draw the line AB and CD whose equations are

$$x+y = 50 \quad \text{...(i)}$$

	A	B
x	0	50
y	50	0

and $\quad 2x+y = 80 \quad$ ...(ii)

	C	D
x	0	40
y	80	0



$\therefore$ The feasible region is ODPAO which is shaded in the figure.

P is the point of intersection of the lines

$$x+y = 50$$
$$2x+y = 80$$

Solving these equations, we get point P (30, 20).

The vertices of the feasible region are (0, 0), D (40, 0), P (30, 20) and A (0, 50).The value of objective function $Z = 10{,}500\, x + 9{,}000\, y$ at these vertices are as follows :

Corner Points	$Z = 10{,}500x + 9{,}000y$
At O (0, 0)	$Z = 0$
At D (40, 0)	$Z = 4{,}20{,}000$
At P (30, 20)	$Z = 4{,}95{,}000$ maximum
At A (0, 50)	$Z = 4{,}50{,}000$

$\therefore$ The maximum profit is 4,95,000 at point P (30, 20). Yes, I agree with the message in the question. **Ans.**

29. Assume that the chances of a patient having a heart attack is 40%. Assuming that a meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chance by 25%. At a time a patient can choose any one of the two options with equal possibilities. It is given that after going through one of the two options, the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga. Interpret the result and state which of the above stated methods is more

beneficial for the patient. **[6]**

Solution : Let E_1 be taking a course of meditation and yoga and E_2 be taking a course of drugs.

A be the patient gets a heart attack.

Here $P(E_1) = \frac{1}{2}; P(E_2) = \frac{1}{2}$

$P(A/E_1) = \frac{70}{100} \times \frac{40}{100}$

$P(A/E_2) = \frac{75}{100} \times \frac{40}{100}$

By Bayes' theorem,

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{70}{100}\times\frac{40}{100}}{\frac{1}{2}\times\frac{70}{100}\times\frac{40}{100}+\frac{1}{2}\times\frac{75}{100}\times\frac{40}{100}} = \frac{14}{29}$$

Now, $P(E_2/A)$

$$= \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{75}{100}\times\frac{40}{100}}{\frac{1}{2}\times\frac{75}{100}\times\frac{40}{100}+\frac{1}{2}\times\frac{70}{100}\times\frac{40}{100}} = \frac{15}{29}$$

Since, $P(E_1/A) < P(E_2/A)$ the course of yoga and meditation is more beneficial for a person having chances of heart attack. **Ans.**

Mathematics 2013 (Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

> **Note :** Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

3. Find the value of b if

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}.$$ **[1]**

Solution : Given, $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

Comparing the corresponding, we get

$a - b = -1$ ...(i)

and $2a - b = 0 \Rightarrow a = \frac{b}{2}$ ...(ii)

From (i) and (ii),

$\frac{b}{2} - b = -1 \Rightarrow \frac{-b}{2} = -1 \Rightarrow b = 2.$ **Ans.**

9. Write the degree of the differential equation $\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0.$ **[1]**

Solution : The degree of the differential equation $\left(\frac{dy}{dx}\right)^4 + 3x\frac{d^2y}{dx^2} = 0$ is 1. **Ans.**

SECTION — B

16. P speaks truth in 70% of the cases and Q in 80% of the cases. In what percent of cases are they likely to agree in stating the same fact ? Do you think, when they agree, means both are speaking truth ? **[4]**

Solution : Probability of P speaking the truth is

$P(P) = \frac{70}{100} = \frac{7}{10} \Rightarrow P(\bar{P}) = 1 - \frac{7}{10} = \frac{3}{10}$

Probability of Q speaking the truth is

$P(Q) = \frac{80}{100} = \frac{8}{10} \Rightarrow P(\bar{Q}) = 1 - \frac{8}{10} = \frac{2}{10}$

Now P and Q are likely to agree with each other in the following mutually exclusive cases.

(i) Both speak the truth

(ii) Both do not speak the truth

By the theorem of total probability,

$\therefore$ Probability that both P and Q agree

$= P(PQ \text{ or } \bar{P}\,\bar{Q})$

$= P(P)\,P(Q) + P(\bar{P})P(\bar{Q})$

$= \frac{7}{10}.\frac{8}{10} + \frac{3}{10}.\frac{2}{10} = \frac{56+6}{100} = \frac{62}{100}$

Hence, they are likely to agree in 62% of the cases.

No, not necessarily when they agree there may be case in which they both does not speak truth.

Ans.

18. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{j}-\hat{k}$, find a vector $\vec{c}$, such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}.\vec{c} = 3$. **[4]**

Solution : Given, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{j}-\hat{k}$

Let $\vec{c} = x_1\hat{i}+x_2\hat{j}+x_3\hat{k}$

$\therefore \quad \vec{a}.\vec{c} = 3$ (Given)

$(\hat{i}+\hat{j}+\hat{k}).(x_1\hat{i}+x_2\hat{j}+x_3\hat{k}) = 3$

$\Rightarrow \quad x_1+x_2+x_3 = 3$ ...(i)

and $\quad \vec{a}\times\vec{c} = \vec{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x_1 & x_2 & x_3 \end{vmatrix} = \hat{j}-\hat{k}$$

$(x_3-x_2)\hat{i}-(x_3-x_1)\hat{j}+(x_2-x_1)\hat{k} = \hat{j}-\hat{k}$

$\Rightarrow \quad x_3-x_2 = 0$ ...(ii)

$x_1-x_3 = 1$ ...(iii)

$x_1-x_2 = 1$ ...(iv)

Solving the equations (i) to (iv), we get

$$x_1 = \frac{5}{3}, x_2 = x_3 = \frac{2}{3}.$$

$$\therefore \quad \vec{c} = \frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$$

$$\therefore \quad \vec{c} = \frac{1}{3}(5\hat{i}+2\hat{j}+2\hat{k}).$$

19. Evaluate : $\int_1^3 [|x-1|+|x-2|+|x-3|]dx$. **[4]**

Solution : Let $I = \int_1^3 [|x-1|+|x-2|+|x-3|]dx$

$$= \int_1^2 [|x-1|+|x-2|+|x-3|]dx + \int_2^3 [|x-1|+|x-2|+|x-3|]dx$$

$$= \int_1^2 [x-1-(x-2)-(x-3)].dx + \int_2^3 [(x-1)+(x-2)-(x-3)].dx$$

$$= \int_1^2 (-x+4)dx + \int_2^3 (x)dx$$

$$= \left[-\frac{x^2}{2}+4x\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3$$

$$= \left[(-2+8)-\left(-\frac{1}{2}+4\right)+\left(\frac{9}{2}-2\right)\right]$$

$$= 6-\frac{7}{2}+\frac{5}{2} = 6-1 = 5.$$ **Ans.**

20. Evaluate : $\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$. **[4]**

Solution : Let $I = \int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$

Putting $x^2 = y$,

$$I = \frac{y+1}{(y+4)(y+25)} = \frac{A}{y+4}+\frac{B}{y+25}$$

$\Rightarrow \quad y+1 = A(y+25)+B(y+4)$

Putting $y = -25$, we get

$$-24 = B(-25+4) \Rightarrow B = \frac{8}{7}$$

and putting $y = -4$, we get

$$-3 = A(-4+25) \Rightarrow A = -\frac{1}{7}.$$

$$\therefore \quad \frac{y+1}{(y+4)(y+25)} = -\frac{1}{7}\left(\frac{1}{y+4}\right)+\frac{8}{7}\left(\frac{1}{y+25}\right)$$

$$\Rightarrow \frac{x^2+1}{(x^2+4)(x^2+25)} = -\frac{1}{7}\left(\frac{1}{x^2+4}\right)+\frac{8}{7}\left(\frac{1}{x^2+25}\right)$$

On integrating both sides w.r.t. x, we get

$$\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx = -\frac{1}{7}\int \frac{dx}{x^2+2^2}+\frac{8}{7}\int \frac{dx}{x^2+5^2}$$

$$= -\frac{1}{7}\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{8}{7}\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)+C$$

$$\left(\text{Using } \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C\right)$$

$$= -\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right)+\frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right)+C.$$ **Ans.**

SECTION — C

28. Show that the differential equation

$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right) = 0$ is homogeneous.

Find the particular solution of this differential equation given that $x = 1$ when $y = \frac{\pi}{2}$. **[6]**

Solution : The given differential equation is

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\left(\frac{y}{x}\right)=0$$

$$\Rightarrow \quad x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)=y\sin\left(\frac{y}{x}\right)-x$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y\sin\left(\frac{y}{x}\right)-x}{x\sin\left(\frac{y}{x}\right)}=\frac{\frac{y}{x}\sin\frac{y}{x}-1}{\sin\frac{y}{x}} \qquad ...(i)$$

which is clearly a homogeneous differentiable equation.

Putting $\frac{y}{x}=v \Rightarrow y=vx$

$$\Rightarrow \quad \frac{dy}{dx}=v+x\frac{dv}{dx}$$

$\therefore$ Eq. (i) becomes

$$v+x\frac{dv}{dx}=\frac{v\sin v-1}{\sin v}=v-\frac{1}{\sin v}$$

$$\Rightarrow \quad x\frac{dv}{dx}=-\frac{1}{\sin v}$$

$$\Rightarrow \quad \sin v\,dv=\frac{-dx}{x}$$

On integrating, we get

$$\Rightarrow \quad \int\sin v\,dv=-\int\frac{1}{x}dx$$

$$\Rightarrow \quad -\cos v=-\log x+C$$

$$\Rightarrow \quad \log x-\cos\left(\frac{y}{x}\right)=C \qquad ...(ii)$$

Given at $x=1$ and $y=\frac{\pi}{2}$

$$\Rightarrow \quad \log 1-\cos\left(\frac{\pi}{2}\right)=C$$

$$\therefore \quad C=0$$

$\therefore$ The particular solution of the given differential equation is

$$\log x-\cos\left(\frac{y}{x}\right)=0. \qquad \textbf{Ans.}$$

29. Find the vector equation of the plane determined by the points A (3, –1, 2), B (5, 2, 4) and C (–1, –1, 6). Also find the distance of point P (6, 5, 9) from this plane. [6]

Solution : The given points are

A (3, – 1, 2); B (5, 2, 4); C (– 1, – 1, 6)

Equation of any plane passes through A (3, – 1, 2) is

$$a(x-3)+b(y+1)+c(z-2)=0 \qquad ...(i)$$

Points B and C lie on it.

$$\therefore \quad 2a+3b+2c=0$$
$$-4a+0b+4c=0$$

Solving these equations, we get

$$\frac{a}{12-0}=\frac{b}{-8-8}=\frac{c}{0+12}$$

$$\Rightarrow \quad \frac{a}{3}=\frac{b}{-4}=\frac{c}{3}$$

$\therefore$ Equation of the required plane is

$$3(x-3)-4(y+1)+3(z-2)=0$$

$$\Rightarrow \quad 3x-4y+3z-19=0 \qquad ...(ii)$$

In vector form, it becomes

$$\vec{r}.(3\hat{i}-4\hat{j}+3\hat{k})-19=0$$

Now distance of P (6, 5, 9) from (ii),

$$\therefore \quad d=\left|\frac{3\times6-4\times5+3\times9-19}{\sqrt{3^2+(-4)^2+3^2}}\right|$$

$$=\left|\frac{18-20+27-19}{\sqrt{34}}\right|=\frac{6}{\sqrt{34}} \qquad \textbf{Ans.}$$

Mathematics 2013 (Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — A

2. Write, unit vector in the direction of the sum of vectors $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}+3\hat{k}$. [1]

Solution : $\vec{a}=2\hat{i}-\hat{j}+2\hat{k},\ \vec{b}=-\hat{i}+\hat{j}+3\hat{k}$

$$\therefore \vec{a}+\vec{b}=(2-1)\hat{i}+(-1+1)\hat{j}+(2+3)\hat{k}=\hat{i}+5\hat{k}$$

$$|\vec{a}+\vec{b}|=\sqrt{1^2+5^2}=\sqrt{26}$$

Therefore, the required unit vector is

$$=\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}=\frac{\hat{i}+5\hat{k}}{\sqrt{26}}. \qquad \textbf{Ans.}$$

4. Write the degree of the differential equation

$$x\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+x^3=0 \quad [1]$$

Solution : The given differential equation is

$$x\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+x^3=0$$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. The power raised to $\frac{d^2y}{dx^2}$ is three.

So, the degree of the given differential equation is 3. **Ans.**

SECTION — B

11. A speaks truth in 75% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact ? Do you think that statement of B is true ? [4]

Solution : Let, the probability that A and B speak truth be P(A) and P(B) respectively.

Therefore, $P(A)=\frac{75}{100}=\frac{3}{4}$ and $P(B)=\frac{90}{100}=\frac{9}{10}$

They can agree in stating the fact when both are speaking the truth or when both are not speaking the truth.

Case 1 : When A is not speaking the truth and B is speaking the truth.

Required probability

$$=(1-P(A))\times P(B)=\left(1-\frac{3}{4}\right)\times\frac{9}{10}=\frac{1}{4}\times\frac{9}{10}=\frac{9}{40}.$$

Case 2 : When A is speaking the truth and B is not speaking the truth.

Required probability

$$=P(A)\times(1-P(B))=\frac{3}{4}\times\left(1-\frac{9}{10}\right)=\frac{3}{4}\times\frac{1}{10}=\frac{3}{40}$$

Therefore, the percent of cases in which they are likely to agree in stating the same fact is equal to

$$\left(\frac{9}{40}+\frac{3}{40}\right)\times 100=30\%.$$ **Ans.**

13. Using vectors, find the area of the triangle ABC with vertices A (1, 2, 3), B (2, – 1, 4) and C (4, 5, – 1). [4]

Solution : The vertices of Δ ABC are A (1, 2, 3), B (2, – 1, 4) and C (4, 5, – 1).

$\therefore \vec{AB}$ = Position vector of B – Position vector of A

$$=(2\hat{i}-\hat{j}+4\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$$
$$=\hat{i}-3\hat{j}+\hat{k}$$

$\vec{AC}$ = Position vector of C – Position vector of A

$$=(4\hat{i}+5\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})$$
$$=3\hat{i}+3\hat{j}-4\hat{k}$$

Now,

$$\vec{AB}\times\vec{AC}=(\hat{i}-3\hat{j}+\hat{k})\times(3\hat{i}+3\hat{j}-4\hat{k})$$

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}=\hat{i}(12-3)-\hat{j}(-4-3)+\hat{k}(3+9)$$

$$=9\hat{i}+7\hat{j}+12\hat{k}$$

$$\therefore |\vec{AB}\times\vec{AC}|=\sqrt{(9)^2+(7)^2+(12)^2}=\sqrt{274}$$

$$\therefore \text{Area of the } \Delta ABC=\frac{1}{2}|\vec{AB}\times\vec{AC}|$$

$$=\frac{1}{2}\sqrt{274} \text{ sq. units.}$$ **Ans.**

14. Evaluate : $\int_2^5[|x-2|+|x-3|+|x-5|]dx$. [4]

Solution : Let, $f(x)=|x-2|+|x-3|+|x-5|$

$$I=\int_2^5 f(x)dx=\int_2^3 f(x)dx+\int_3^5 f(x)dx$$

$$\Rightarrow I=\int_2^3(x-2+3-x+5-x)dx+\int_3^5(x-2+x-3+5-x)dx$$

$$\Rightarrow I=\int_2^3(6-x)dx+\int_3^5 x\,dx=\left[6x-\frac{x^2}{2}\right]_2^3+\left[\frac{x^2}{2}\right]_3^5$$

$$=\left[18-\frac{9}{2}-12+2\right]+\left[\frac{25}{2}-\frac{9}{2}\right]=\frac{23}{2}.$$ **Ans.**

15. Evaluate : $\int\frac{2x^2+1}{x^2(x^2+4)}dx$. [4]

Solution :

$$I=\int\frac{2x^2+1}{x^2(x^2+4)}dx=\int\frac{2x^2+8-7}{x^2(x^2+4)}dx$$

$$=\int\frac{2(x^2+4)-7}{x^2(x^2+4)}dx$$

[Multiplying & dividing by 4 and then adding and subtracting x^2 in 2^{nd} Integral]

$$= \int \frac{2}{x^2}dx - 7\int \frac{1}{x^2(x^2+4)}dx$$

$$\Rightarrow \quad I = \int \frac{2}{x^2}dx - \frac{7}{4}\int \frac{(x^2+4)-x^2}{x^2(x^2+4)}dx$$

$$= \int \frac{2}{x^2}dx - \frac{7}{4}\int \frac{1}{x^2}dx + \frac{7}{4}\int \frac{1}{(x^2+4)}dx$$

$$\Rightarrow \quad I = \frac{1}{4}\int \frac{1}{x^2}dx + \frac{7}{4}\int \frac{1}{(x^2+4)}dx$$

$$= \frac{1}{4}\left(-\frac{1}{x}\right) + \frac{7}{4} \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$$

$$= -\frac{1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C. \qquad \textbf{Ans.}$$

SECTION — C

25. Find the coordinates of the point where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane, passing through the points (2, 2, 1), (3, 0, 1) and (4, – 1, 0). [6]

Solution : The equation of the straight line passing through the point (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{2-3} = \frac{y-(-4)}{-3-(-4)} = \frac{z-(-5)}{1-(-5)}$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda \text{ (say)} \qquad ...(i)$$

The coordinates of any point on line (i) is

$(-\lambda+3, \lambda-4, 6\lambda-5)$.

We know that, the equation of the plane passing through three points

$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

So, the equation of the plane passing through the points (2, 2, 1), (3, 0, 1) and (4, – 1, 0) is

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(2-0)-(y-2)(-1-0)+(z-1)(-3+4) = 0$$

$$\Rightarrow \quad 2x-4+y-2+z-1 = 0$$

$$\Rightarrow \quad 2x+y+z-7 = 0 \qquad ..(ii)$$

If the point $(-\lambda+3, \lambda-4, 6\lambda-5)$ lies on the plane (ii), then

$$2(-\lambda+3)+(\lambda-4)+(6\lambda-5)-7 = 0$$

$$\Rightarrow \quad 5\lambda - 10 = 0$$

$$\Rightarrow \quad \lambda = 2$$

Putting $\lambda = 2$ in $(-\lambda+3, \lambda-4, 6\lambda-5)$, we have $(-2+3, 2-4, 6\times 2-5) = (1, -2, 7)$

Thus, the coordinates of the point where the line (i) crosses the plane (ii) is (1, – 2, 7). **Ans.**

26. Show that the differential equation $(xe^{y/x}+y)\,dx = x\,dy$ is homogeneous. Find the particular solution of this differential equation, given that $x = 1$ when $y = 1$. [6]

Solution : The given differential equation is

$$(xe^{\frac{y}{x}}+y)dx = xdy$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{xe^{\frac{y}{x}}+y}{x} = f(x,y) \qquad ...(i)$$

Replacing x by λx and y by λy in (i), we have

$$f(\lambda x, \lambda y) = \frac{(\lambda x)e^{\lambda x/\lambda y}+\lambda y}{\lambda x}$$

$$= \lambda^0\left(\frac{xe^{\frac{y}{x}}+y}{x}\right) = \lambda^0 f(x,y)$$

Therefore, the given differential equation is homogeneous whose degree is 0.

$$\frac{dy}{dx} = \frac{xe^{\frac{y}{x}}+y}{x}$$

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = e^v + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = e^v \Rightarrow e^{-v}\,dv = \frac{dv}{dx}$$

Integrating both sides

$$\int e^{-v}dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad -e^{-v} = \log|x| + c$$

$$\Rightarrow \quad -e^{-y/x} = \log|x| + c$$

It is given that at $x = 1, y = 1$

$$-e^{-1} = \log 1 + c \Rightarrow c = \frac{1}{e}$$

$\therefore$ Particular solution is

$$-e^{-y/x} = \log|x| - \frac{1}{e}$$

$$\Rightarrow \quad -\frac{y}{x} = \log\left(\frac{1}{e} - \log|x|\right)$$

$$= \log\left(\frac{1-e\log|x|}{e}\right)$$

$$= \log(1-e\log|x|) - \log e$$

$$\Rightarrow \quad y = x - x\log(1-e\log|x|). \qquad \textbf{Ans.}$$

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